# RAMIFICATION OF PRIMES IN FIELDS OF MODULI OF THREE-POINT COVERS

Andrew Obus

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David Harbater
Supervisor of Dissertation

Tony Pantev
Graduate Group Chairperson

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ABSTRACT

RAMIFICATION OF PRIMES IN FIELDS OF MODULI OF THREE-POINT COVERS

Andrew Obus

David Harbater, Advisor

We examine in detail the stable reduction of three-point $G$-Galois covers of the projective line over a complete discrete valuation field of mixed characteristic $(0, p)$, where $G$ has a cyclic $p$-Sylow subgroup. In particular, we obtain results about ramification of primes in the minimal field of definition of the stable model of such a cover, under certain additional assumptions on $G$ (one such sufficient, but not necessary set of assumptions is that $G$ is solvable and $p \neq 2$). This has the following consequence: Suppose $f : Y \to \mathbb{P}^1$ is a three-point $G$-Galois cover defined over $\mathbb{C}$, where $G$ has a cyclic $p$-Sylow subgroup of order $p^n$, and these additional assumptions on $G$ are satisfied. Then the $n$th higher ramification groups above $p$ for the upper numbering for the extension $K/\mathbb{Q}$ vanish, where $K$ is the field of moduli of $f$.

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# Chapter 1

# Introduction

## 1.1 Overview

This thesis focuses on understanding how primes of $\mathbb{Q}$ ramify in the field of moduli of three-point Galois covers of the Riemann sphere. I generalize a result of Wewers about ramification of primes $p$ where $p$ divides the order of the Galois group and the $p$-Sylow subgroup of the Galois group is cyclic (Theorem 1.4).

Let $X$ be the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, and let $f : Y \to X$ be a finite branched cover of Riemann surfaces. By Serre's GAGA principle ([GAGA]), $Y$ is isomorphic to an algebraic variety and we can take $f$ to be an algebraic, regular map. By a theorem of Grothendieck, if the branch points of $f$ are $\overline{\mathbb{Q}}$-rational (for example, if the cover is branched at three points, which we can always take to be 0, 1, and $\infty$—such a cover is called a *three-point cover*), then the equations of the cover $f$ can themselves be defined over $\overline{\mathbb{Q}}$ (in fact, over some number field). Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = G_{\mathbb{Q}}$. Since $X$ is defined over $\mathbb{Q}$, then $\sigma$ acts on the set of branched covers of $X$ by acting on the coefficients of the defining equations. We write $f^\sigma : Y^\sigma \to X^\sigma$ for the cover thus obtained. If $f : Y \to X$ is Galois with group $G$, let $\Gamma^{in} \subset G_{\mathbb{Q}}$ be the subgroup consisting of those $\sigma$ which preserve the isomorphism class of $f$ as well as the $G$-action. That is, $\Gamma^{in}$ consists of those elements $\sigma$ of $G_{\mathbb{Q}}$ such that there is an isomorphism $\phi : Y \to Y^\sigma$, commuting with the action of $G$, which makes the following diagram commute:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y^\sigma \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^\sigma} \\ X & = & X^\sigma \end{array}$$

The fixed field $\overline{\mathbb{Q}}^{\Gamma^{in}}$ is known as the *field of moduli* of $f$ (as a $G$-cover). It is the intersection of all the fields of definition of $f$ as a $G$-cover (i.e., those fields of definition $K$ of $f$ such that the action of $G$ can also be written in terms of polynomials with coefficients in $K$); see [CH85, Proposition 2.7]. Occasionally, we will also discuss the fields of moduli and fields of definition of branched covers as *mere covers*. This means (for the field of moduli) that $\phi$ need not commute with the $G$-action, and (for the field of definition) that the $G$-action need not be defined over the field in question.

Now, since a branched $G$-Galois cover $f : Y \to X$ of the Riemann sphere is given entirely in terms of combinatorial data (the branch locus $C$, the Galois group $G$, and the monodromy action of $\pi_1(X \backslash C)$ on $Y$), it is reasonable to try to draw inferences about the field of moduli of $f$ based on these data. However, not much is known about this, and this is the goal toward which we work.

The problem of determining the field of moduli of three-point covers has applications towards understanding the *fundamental exact sequence*

$$1 \to \pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}) \to \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}) \to G_{\mathbb{Q}} \to 1,$$

where $\pi_1$ is the étale fundamental group functor. This is a very interesting and not fully understood object (note that a complete understanding would yield a complete understanding of $G_{\mathbb{Q}}$). The exact sequence gives rise to an outer action of $G_{\mathbb{Q}}$ on $\Pi = \pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})$. This outer action would be particularly interesting to understand. Knowing about fields of moduli aids our understanding as follows: Say the $G$-Galois cover $f$ corresponds to the normal subgroup $N \subset \Pi$, so that $\Pi/N \cong G$. Then the group $\Gamma^{in}$ consists exactly of those elements of $G_{\mathbb{Q}}$ whose outer action on $\Pi$ both preserves $N$ and descends to an inner action on $\Pi/N \cong G$.

**Remark 1.1.** It might seem that restricting to the case of three-point covers of $\mathbb{P}^1$ is a very drastic restriction when studying curves. In fact, according to Belyi's theorem ([Bel79, Theorem 4]), any curve defined over $\overline{\mathbb{Q}}$ has a map to $\mathbb{P}^1$ branched at three points. So while the case of three-point covers certainly does not include all covers of curves, it does include *some* cover $f : Y \to X$ for each curve $Y$ defined over $\overline{\mathbb{Q}}$.

## 1.2 Main result

One of the first major results in this direction is due to Beckmann:

**Theorem 1.2** ([Bec89])**.** *Let $f : Y \to X$ be a branched $G$-Galois cover of the Riemann sphere, with branch points defined over $\overline{\mathbb{Q}}$. Then $p \in \mathbb{Q}$ can be ramified in the field of moduli of $f$ as a $G$-cover only if $p$ is ramified in the field of definition of a branch point,* *or $p \mid |G|$, or there is a collision of branch points modulo some prime dividing $p$. In particular, if $f$ is a three-point cover and if $p \nmid |G|$, then $p$ is unramified in the field of moduli of $f$.*

This result was partially generalized by Wewers:

**Theorem 1.3** ([Wew03b])**.** *Let $f : Y \to X$ be a three-point $G$-Galois cover of the Riemann sphere, and suppose that $p$* exactly *divides $|G|$. Then $p$ is tamely ramified in the field of moduli of $f$ as a $G$-cover.*

In fact, Wewers shows somewhat more, in that he computes the index of tame ramification of $p$ in the field of moduli in terms of some invariants of $f$.

To state my main theorem, which is a further generalization, we will need some group theory. Call a finite group $G$ *$p$-solvable* if its only simple composition factors with order divisible by $p$ are isomorphic to $\mathbb{Z}/p$. Clearly, any solvable group is $p$-solvable. If $H \subset G$, we write $N_G(H)$ for the normalizer of $H$ in $G$ and $Z_G(H)$ for the centralizer of $H$ in $G$.

My main result is:

**Theorem 1.4.** *Let $f : Y \to X$ be a three-point $G$-Galois cover of the Riemann sphere, and suppose that a $p$-Sylow subgroup $P \subset G$ is cyclic of order $p^n$. Let $m = |N_G(P)/Z_G(P)|$. Let $K/\mathbb{Q}$ be the field of moduli of $f$. Then the $n$th higher ramification groups for the upper numbering of $K/\mathbb{Q}$ vanish in either of the following cases:*

*(i) $G$ is $p$-solvable and either $p \neq 2$ or $G \cong \mathbb{Z}/2^n$.*

*(ii) $m = 2$, provided either that at least two of the three branch points have prime-to-$p$*

*branching index, or that one of the three branch points has prime-to-p branching index and $p \neq 3, 5$.*

**Remark 1.5.** (i) Note that Beckmann's and Wewers's theorems cover the cases $n = 0, 1$ in the notation above.

(ii) Any group that satisfies $m = 1$ is $p$-solvable by a theorem of Burnside (Lemma 2.2). We will show (Theorem 2.7) that if $G$ has a cyclic $p$-Sylow subgroup and is not $p$-solvable, it must have a simple composition factor with order divisible by $p^n$. There seem to be limited examples of simple groups with cyclic $p$-Sylow subgroups of order greater than $p$. For instance, there are no sporadic groups or alternating groups. There are some of the form $PSL_r(q)$. There is also the Suzuki group $Sz(32)$. Other than these examples, there are none listed in [ATLAS], but this certainly does not preclude their existence. Furthermore, many of the examples that do exist satisfy $m = 2$ (for instance, $PSL_2(q)$, where $p^n$ exactly divides $q^2 - 1$).

(iii) I fully expect Theorem 1.4 to hold in the case $m = 2$, even without any assumptions on $p$ or on the number of points with prime-to-$p$ branching index. In fact, I believe I have a proof for $m = 2$, $p = 5$ in the full generality of Theorem 1.4, save one exceptional case, but for reasons of time I have not been able to include it. See Question 5.1.

(iv) If $G$ has a cyclic 2-Sylow subgroup, then it follows from Lemmas 2.1 and 2.2 that $G \cong N \rtimes \mathbb{Z}/2^n$, where $2 \nmid |N|$ (in particular, $G$ is 2-solvable). I expect Theorem 1.4 to hold in this case as well.

## 1.3 Chapter-by-chapter summary and walkthrough

In Chapter 2, we provide the background that is needed for the proof of Theorem 1.4. We start with §2.1, which proves structure theorems about finite groups with cyclic $p$-Sylow groups. The material in §2.2 and §2.4 is well-known. The basic material in §2.3 is also well-known, but we prove several lemmas that will come in handy in §3.1 and §4.3. The next few sections introduce the major players in the proof of Theorem 1.4. Specifically, §2.5 introduces the *stable model*. The results in this section come from [Ray99]. In §2.6, the construction of the *auxiliary cover* $f^{aux}$ corresponding to a cover $f$ is described, along with why it is useful. In particular, Lemma 2.28 gives a connection between the fields of definition of $f^{aux}$ and those of $f$. Then, §2.7 introduces *deformation data*, which played a large role in the proof of Theorem 1.3. Wewers was dealing with covers where the order of the Galois group was divisible by at most one factor of $p$, and here we extend the idea of deformation data to our more general context. In addition to giving definitions and constructions, we prove many formulas that are used in the proofs of the main theorems.

In Chapter 3, we prove several results which collectively go under the heading of "Vanishing Cycles Formulas." These formulas give us detailed information about the irreducible components of the stable reduction of a cover $f$. These results are used in §3.2 to place limits on how complicated the stable reduction of $f$ can be.

In Chapter 4, we prove the main Theorem 1.4. We start by outlining the proof. The proof splits into three cases, corresponding to §4.1, §4.2, and §4.3. The proof of each case depends heavily on the results of Chapters 2 and 3.

In Chapter 5, we discuss some of the questions arising from this thesis.

## 1.4 Notations and conventions

The following notations will be used throughout the thesis: The letter $p$ always represents a prime number. If $G$ is a group and $H$ a subgroup, we write $H \leq G$. We denote by $N_G(H)$ the normalizer of $H$ in $G$ and by $Z_G(H)$ the centralizer of $H$ in $G$. The order of $G$ is written $|G|$. If $G$ has a cyclic $p$-Sylow subgroup $P$, and $p$ is understood, we write $m_G = |N_G(P)/Z_G(P)|$. If a group $G$ has a unique cyclic subgroup of order $p^i$, we will occasionally abuse notation and write $\mathbb{Z}/p^i$ for this subgroup.

If $K$ is a field, $\overline{K}$ is its algebraic closure. We write $G_K$ for the absolute Galois group of $K$. If $H \leq G_K$, write $K^H$ for the fixed field of $H$ in $\overline{K}$. If $K$ discretely valued, then $K^{ur}$ is the *completion* of the maximal unramified algebraic extension of $K$. If $K$ is a field of characteristic 0 (resp. characteristic $p$), then for all $j \in \mathbb{Z}_{>0}$ (resp. those $j$ which are prime to $p$), we fix a compatible system of primitive $j$th roots of unity $\zeta_j$. "Compatible" means that $\zeta_{ij}^i = \zeta_j$ for all $i, j$. The notations $\mu_p, \alpha_p$ denote group schemes.

If $x$ is a scheme-theoretic point of a scheme $X$, then $\mathcal{O}_{X,x}$ is the local ring of $x$ on $X$. For any local ring $R$, $\hat{R}$ is the completion of $R$ with respect to its maximal ideal. If $R$ is any ring with a non-archimedean absolute value $|\cdot|$, then the ring $R\{T\}$ is the ring of power series $\sum_{i=0}^{\infty} c_i T^i$ such that $\lim_{i\to\infty} |c_i| = 0$. If $R$ is a discrete valuation ring with fraction field $K$ of characteristic 0 and residue field $k$ of characteristic $p$, we normalize the absolute value on $K$ and on any subring of $K$ so that $|p| = 1/p$.

A *branched cover* $f : Y \to X$ of smooth proper curves is a finite surjective morphism. All branched covers are assumed to be geometrically connected, unless explicitly noted otherwise. If $f$ is of degree $d$ and $G$ is a finite group of order $d$ with $G \cong \mathrm{Aut}(Y/X)$, then $f$ is called a *Galois cover with (Galois) group $G$*. If we choose an isomorphism $i : G \to \mathrm{Aut}(Y/X)$, then the datum $(f, i)$ is called a *$G$-Galois cover* (or just a *$G$-cover*, for short). We will usually suppress the isomorphism $i$, and speak of $f$ as a $G$-cover.

Suppose $f : Y \to X$ is a $G$-cover of smooth curves, and $K$ is a field of definition for $X$. Then the *field of moduli of $f$ relative to $K$ (as a $G$-cover)* is the fixed field in $\overline{K}/K$ of $\Gamma^{in} \subset G_K$, where $\Gamma^{in} = \{\sigma \in G_K | f^\sigma \cong f(\text{as } G\text{-covers})\}$ (see §1.1). If $X$ is $\mathbb{P}^1$, then the *field of moduli of $f$* means the field of moduli of $f$ relative to $\mathbb{Q}$. Unless otherwise stated, a field of definition (or moduli) of $f$ (or the stable model of $f$) means a field of definition (or moduli) *as a $G$-cover* (see §1.1). If we do not want to consider the $G$-action, we will always explicitly say the field of definition (or moduli) *as a mere cover* (this happens only in Chapter 4).

The *ramification index* of a point $y \in Y$ such that $f(y) = x$ is the ramification index of the extension of local rings $\hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{Y,y}$. If $f$ is Galois, then the *branching index* of a closed point $x \in X$ is the ramification index of any point $y$ in the fiber of $f$ over $x$. If the ramification index of $y$ (resp. the branching index of $x$) is greater than 1, then $y$ (resp. $x$) is called a *ramification point* (resp. *branch point*).

Let $f : Y \to X$ be any morphism of schemes and suppose $H$ is a finite group with $H \hookrightarrow \mathrm{Aut}(Y/X)$. If $G$ is a finite group containing $H$, then there is a map $\mathrm{Ind}_H^G Y \to X$, where $\mathrm{Ind}_H^G Y$ is a disjoint union of $[G : H]$ copies of $Y$, indexed by the left cosets of $H$ in $G$. The group $G$ acts on $\mathrm{Ind}_H^G Y$, and the stabilizer of each copy of $Y$ in $\mathrm{Ind}_H^G Y$ is a conjugate of $H$.

If $X$ is a scheme, then $p_a(X)$ represents its arithmetic genus.

For any real number $r$, $\lfloor r \rfloor$ is the greatest integer less than or equal to $r$. Also, $\langle r \rangle = r - \lfloor r \rfloor$. By convention, all integers are prime to 0 and 0 does not divide any integer.

The phrase "$a := b$" means that $a$ is defined to be equal to $b$.

# Chapter 2

# Background Material

## 2.1 Finite groups with cyclic $p$-Sylow subgroups

In this section, we prove structure theorems about finite groups with *cyclic* $p$-Sylow subgroups. The main results are Corollary 2.4 and Theorem 2.7. Throughout §2.1, $G$ is a finite group with a cyclic $p$-Sylow subgroup $P$ of order $p^n$. Recall that $m_G = |N_G(P)/Z_G(P)|$. We will often abbreviate $m_G$ to $m$ when $G$ is understood.

**Lemma 2.1.** *Let $Q \leq P$ have order $p$. Then if $g \in N_G(P)$ acts trivially on $Q$ by conjugation, it acts trivially on $P$. Thus $N_G(P)/Z_G(P) \hookrightarrow Aut(Q)$, so $m|(p-1)$.*

*Proof.* We know $\mathrm{Aut}(P) \cong (\mathbb{Z}/p^n)^\times$, which has order $p^{n-1}(p-1)$, with a unique maximal prime-to-$p$ subgroup $C$ of order $p-1$. Let $g \in N_G(P)$, and suppose that the image $\overline{g}$ of $g$ in $N_G(P)/Z_G(P) \subseteq \mathrm{Aut}(P)$ acts trivially on $Q$. Since

$$(\mathbb{Z}/p^n)^\times \cong \mathrm{Aut}(P) \twoheadrightarrow \mathrm{Aut}(Q) \cong (\mathbb{Z}/p)^\times,$$

has $p$-group kernel we know that $\overline{g}$ has $p$-power order. If $\overline{g}$ is not trivial, then $g \notin P$, and the subgroup $\langle g, P\rangle$ of $G$ has a non-cyclic $p$-Sylow subgroup. This is impossible, so $\overline{g}$ is trivial, and $g$ acts trivially on $P$. □

We state a theorem of Burnside that will be useful in proving Corollary 2.4:

**Lemma 2.2** ([Zas56], Theorem 4, p. 169)**.** *Let* $\Gamma$ *be a finite group, with a* $p$*-Sylow subgroup* $\Pi$*. Then, if* $N_\Gamma(\Pi) = Z_\Gamma(\Pi)$*, the group* $\Gamma$ *can be written as an extension*

$$1 \to \Sigma \to \Gamma \to \Pi' \to 1,$$

*where* $\Pi \le \Gamma$ *maps isomorphically onto* $\Pi'$*.*

Recall that a group $G$ is *$p$-solvable* if its only simple composition factors with order divisible by $p$ are isomorphic to $\mathbb{Z}/p$.

**Proposition 2.3.** *Suppose that a finite group* $G'$ *has a normal subgroup* $M$ *of order* $p$ *contained in a cyclic* $p$*-Sylow subgroup* $Q$*, and no nontrivial normal subgroups of prime-to-*$p$ *order. Then* $G' \cong Q \rtimes \mathbb{Z}/m_{G'}$*. In particular,* $G'$ *is solvable.*

*Proof.* Consider the centralizer $C := Z_{G'}(M)$. Now, $Q$ is clearly a $p$-Sylow subgroup of $C$. We claim that $N_C(Q) = Z_C(Q)$.

To prove the claim, say $g \in N_C(Q)$. If $g$ has $p$-power order, then $g \in Q$ (otherwise the $p$-Sylow subgroup of $C$ would not be cyclic), thus $g \in Z_C(Q)$. If $g$ has prime-to-$p$ order, then $g$ induces an element $h$ of prime-to-$p$ order in $\mathrm{Aut}(Q)$. But the prime-to-$p$ part of $\mathrm{Aut}(Q)$ is canonically isomorphic to $\mathrm{Aut}(M)$ via restriction, and $g$ centralizes $M$, being in $C$. Thus $g$ centralizes $Q$. Lastly, if the order of $g$ is divisible by $p$ but is not a power of $p$, then $g^a$ is of $p$-power order for some $a$ prime to $p$. Thus $g$ induces an element $h$ of order $a$ in $\mathrm{Aut}(Q)$. Again, since $\mathrm{Aut}(Q) \cong \mathrm{Aut}(M)$ via restriction and $g$ centralizes $M$, it follows that $a = 1$ and we are reduced to the previous case. So the claim is proved.

Since $N_C(Q) = Z_C(Q)$, Lemma 2.2 shows that $C$ can be written as an extension

$$1 \to S \to C \to \overline{Q} \to 1,$$

where $Q \subseteq C$ maps isomorphically onto $\overline{Q}$. The group $S$, being the maximal normal prime-to-$p$ subgroup of $C$, is characteristic in $C$. Since $C$ is normal in $G'$ (it is the centralizer of the normal subgroup $M$), $S$ is normal in $G'$. But by assumption, $G'$ has no nontrivial normal subgroups of prime-to-$p$ order, so $S$ is trivial and $C = Q$. Again, since $C$ is normal in $G'$, then $G'$ is of the form $Q \rtimes T$, where $T$ is prime-to-$p$, by the Schur-Zassenhaus theorem. The conjugation action of $T$ on $Q$ must be faithful, since if there were a kernel, the kernel would be a nontrivial prime-to-$p$ normal subgroup of $G'$, contradicting the assumption that $G'$ has none. Since the subgroup of $\mathrm{Aut}(Q)$ induced by this action is cyclic of order $m_{G'}$, we have $T \cong \mathbb{Z}/m_{G'}$. □

**Corollary 2.4.** *Suppose $G$ is $p$-solvable. Let $N$ be the maximal prime-to-$p$ normal subgroup of $G$. Then $G/N \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$, where the conjugation action of $\mathbb{Z}/m_G$ on $\mathbb{Z}/p^n$ is faithful.*

*Proof.* Consider a minimal normal subgroup $M$ of the group $G' := G/N$. We know that $M$ is a direct product of isomorphic simple groups, each with order divisible by $p$. Since a $p$-Sylow subgroup of $G$ (hence $G'$) is cyclic, a $p$-Sylow subgroup of $M$ is as well, and thus $M$ must in fact be a simple group. Since $G$ (hence $G'$) is $p$-solvable, $M \cong \mathbb{Z}/p$. Then $M$

is normal in $G'$, and is contained in some subgroup $Q \cong \mathbb{Z}/p^n$. By construction, $G'$ has no nontrivial normal subgroups of prime-to-$p$ order. So we conclude by Proposition 2.3.

□

**Corollary 2.5.** *If $G$ has a normal subgroup of order $p$, it is $p$-solvable.*

*Proof.* Let $N$ be the maximal normal prime-to-$p$ subgroup of $G$. Then $G/N$ still has a normal subgroup of order $p$, and no nontrivial normal subgroups of prime-to-$p$ order. By Corollary 2.4 $G/N \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. Since $N$ is prime-to-$p$, this means that $G$ is $p$-solvable. □

Before we prove the main theorem of this section, we prove a lemma. Our proof depends on the classification of finite simple groups.

**Lemma 2.6.** *Let $S$ be a nonabelian finite simple group with a cyclic $p$-Sylow subgroup. Then any element $\overline{x} \in Out(S)$ with order $p$ lifts to an automorphism $x \in Aut(S)$ with order $p$.*

*Proof.* All facts about finite simple groups used in this proof that are not clear from the definitions or otherwise cited can be found in [ATLAS].

First note that $p \neq 2$. This is because $m_S|(p-1)$ (Lemma 2.1), so if $p = 2$, we would have $m_S = 1$. By Lemma 2.2 and the simplicity of $S$, we would have that $S$ is equal to its (cyclic) $p$-Sylow subgroup, which is a contradiction. Note also that no primes other than 2 divide the order of the outer automorphism group of any alternating or sporadic group. So we may assume that $S$ is of Lie type.

We first show that $p$ does not divide the order $d$ of the diagonal automorphism group or $g$ of the graph automorphism group of $S$. The only simple groups $S$ of Lie type for which an odd prime divides $g$ are those of the form $O_8^+(q)$. In this case $3|g$. But $O_8^+(q)$ contains $(O_4^+(q))^2$ in block form, and the order of $O_4^+(q)$ is $\frac{1}{(4,q^2-1)}(q^2(q^2-1)^2)$. This is divisible by 3, so $O_8^+(q)$ contains the group $\mathbb{Z}/3 \times \mathbb{Z}/3$, and does not have a cyclic 3-Sylow subgroup. The simple groups $S$ of Lie type for which an odd prime $p$ divides $d$ are the following:

1. $PSL_n(q)$, for $p|(n, q-1)$.

2. $PSU_n(q^2)$, for $p|(n, q+1)$.

3. $E_6(q)$, for $p = 3$ and $3|(q-1)$.

4. ${}^2E_6(q^2)$, $p = 3$ and $3|(q+1)$.

Now, $PSL_n(q)$ contains a split maximal torus $((\mathbb{Z}/q)^\times)^{n-1}$. Since $p|(q-1)$, this group contains $(\mathbb{Z}/p)^{n-1}$ which is not cyclic, as $p|n$ and $p \neq 2$. So a $p$-Sylow subgroup of $PSL_n(q)$ is not cyclic. The diagonal matrices in $PSU_n(q^2)$ form the group $(\mathbb{Z}/(q+1))^{n-1}$, which also contains a non-cyclic $p$-group. The group $E_6(q)$ has a split maximal torus $((\mathbb{Z}/q)^\times)^6$ ([Hum75, §35]), and thus contains a non-cyclic 3-group. Lastly, ${}^2E_6(q^2)$ is constructed as a subgroup of $E_6(q^2)$. When $q \equiv -1 \pmod 3$, the ratio $|E_6(q^2)|/|{}^2E_6(q^2)|$ is not divisible by 3, so a 3-Sylow subgroup of ${}^2E_6(q^2)$ is isomorphic to one of $E_6(q^2)$, which we already know is not cyclic.

So we know that if there exists an element $\overline{x} \in \mathrm{Out}(S)$ of order $p$, we must have that $p$ divides $f$, the order of the group of field automorphisms. Also, since the group of field automorphisms is cyclic and $p$ does not divide $d$ or $g$, a $p$-Sylow subgroup of $\mathrm{Out}(S)$ is cyclic. This means that all elements of order $p$ in $\mathrm{Out}(S)$ are conjugate in $\mathrm{Out}(S)$. Now, there exists an automorphism $\alpha$ in $\mathrm{Aut}(S)$ which has order $p$ and is not inner. Namely, we view $S$ as the $\mathbb{F}_q$-points of some $\mathbb{Z}$-scheme, where $q = \wp^f$ for some prime $\wp$, and we act on these points by the $(f/p)$th power of the Frobenius at $\wp$. Let $\overline{\alpha}$ be the image of $\alpha$ in $\mathrm{Out}(S)$. Since $\overline{\alpha}$ is conjugate to $\overline{x}$ in $\mathrm{Out}(S)$, there exists some $x$ conjugate to $\alpha$ in $\mathrm{Aut}(S)$ such that $\overline{x}$ is the image of $x$ in $\mathrm{Out}(S)$. Then $x$ must have order $p$. It is the automorphism we seek. □

The main theorem we wish to prove in this section essentially states that a finite group with a cyclic $p$-Sylow subgroup is either $p$-solvable or "as far from $p$-solvable as possible."

**Theorem 2.7.** *Let $G$ be a finite group with a cyclic $p$-Sylow subgroup $P$ of order $p^n$. Then one of the following two statements is true:*

- *$G$ is $p$-solvable.*
- *$G$ has a simple composition factor $S$ with $p^n \mid |S|$.*

*Proof.* Clearly we may replace $G$ by $G/N$, where $N$ is the maximal prime-to-$p$ normal subgroup of $G$. So assume that any nontrivial normal subgroup of $G$ has order divisible by $p$. Let $S$ be a minimal normal subgroup of $G$. Then $S$ is a direct product of isomorphic simple groups. Since $G$ has cyclic $p$-Sylow subgroup, and no normal subgroups of prime-to-$p$ order, we see that $S$ is a simple group with $p \mid |S|$. If $S \cong \mathbb{Z}/p$, Corollary 2.5 shows that $G$ is $p$-solvable. So assume, for a contradiction, that $p^n \nmid |S|$ and $S \not\cong \mathbb{Z}/p$. Then $G/S$ contains a subgroup of order $p$. Let $H$ be the inverse image of this subgroup in $G$.

It follows that $H$ is an extension of the form

$$1 \to S \to H \to H/S \cong \mathbb{Z}/p \to 1. \qquad (2.1.1)$$

We claim that $H$ cannot have a cyclic $p$-Sylow subgroup, thus obtaining the desired contradiction.

To prove our claim, we show that $H$ is in fact a semidirect product $S \rtimes H/S$, i.e., we can lift $H/S$ to a subgroup of $H$. Let $\overline{x}$ be a generator of $H/S$. We need to find a lift $x$ of $\overline{x}$ which has order $p$. It suffices to find $x$ lifting $\overline{x}$ such that conjugation by $x^p$ on $S$ is the trivial isomorphism, as $S$ is center-free. In other words, we seek an automorphism of $S$ of order $p$ which lifts the outer automorphism $\phi_{\overline{x}}$ of order $p$ given by $\overline{x}$. This automorphism is provided by Lemma 2.6, finishing the proof. □

## 2.2 Covers in characteristic zero

Let $C = \{p_0, \ldots, p_n\}$ be a nonempty finite set of closed points of $X = \mathbb{P}^1_{\mathbb{C}}$. Consider $U = X \backslash C$ as a complex manifold. If $\xi$ is a base point, the fundamental group $\pi_1(U, \xi)$ can be generated by elements $\gamma_i$, $0 \leq i \leq n$, where each $\gamma_i$ loops once counterclockwise from $\xi$ around $p_i$ and where $\prod_{i=0}^{n} \gamma_i = 1$. Then $\pi_1(U, \xi)$ is presented by the $\gamma_i$ with this relation. The ordered $n+1$-tuple given by the $\gamma_i$ is called a *standard homotopy basis* for $U$ at $\xi$.

Fix a standard homotopy basis $\gamma_i$ for $U$ at $\xi$. If $f : Y \to X$ is a $G$-Galois branched cover, étale over $U$, then $f$ corresponds to a unique open normal subgroup $N \leq \pi_1(U)$ such that $\pi_1(U)/N \cong G$. This isomorphism is fixed once we pick a base point $\zeta$ of $Y$ above

$\xi$. Fix $\zeta$, and write $\overline{\gamma}_i$ for the image of $\gamma_i$ in $G$. Then the $n+1$-tuple $(\overline{\gamma}_0, \ldots, \overline{\gamma}_n) \in G^{n+1}$ is called the *description* of the pointed cover $f$. The $\overline{\gamma}_i$ generate $G$, and their product is the identity. The order of each $\overline{\gamma}_i$ is the branching index of the point $p_i$ in $f$. If we consider $f$ as a non-pointed cover, then the description is defined only up to uniform conjugation by an element in $G$. By the Riemann Existence Theorem, this correspondence is in fact a bijection between algebraic branched covers and $n+1$-tuples $g_i$ of elements of $G$ such that $\prod_{i=0}^{n} g_i = 1$, up to uniform conjugation by an element of $G$. See also [CH85, p. 822].

Let $K$ be an algebraically closed field of characteristic 0. Let $X_K = \mathbb{P}^1_K$. Fix embeddings $i_K : \overline{\mathbb{Q}} \hookrightarrow K$ and $i_{\mathbb{C}} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Suppose the finite set $C = \{p_0, \ldots, p_n\}$ consists of $\overline{\mathbb{Q}}$-rational points, and that $\xi$ is $\overline{\mathbb{Q}}$-rational. View the set $C \cup \{\xi\}$ as a subset of $\mathbb{P}^1_K$ via $i_K$. Let $U_K = X_K \backslash C$. Then, by [Sza09, Theorem 4.6.10], the embeddings $i_K$ and $i_{\mathbb{C}}$ give rise to an isomorphism of étale fundamental groups $\pi_1^{\text{ét}}(U_K, \xi) \xrightarrow{\sim} \pi_1^{\text{ét}}(U, \xi)$. In particular, this yields a bijection between $G$-Galois branched covers of $X_K$, étale over $U_K$, and $G$-Galois branched covers of $X$, étale over $U$. Thus $G$-Galois branched covers of $X_K$, étale over $U_K$, are in natural bijection with $n+1$-tuples $g_i$ of elements of $G$ such that $\prod_{i=0}^{n} g_i = 1$, up to uniform conjugation by an element of $G$.

## 2.3 Basic facts about (wild) ramification

We state here some facts from [Ser79, IV] and derive some consequences. Let $K$ be a complete discrete valuation field with algebraically closed residue field $k$ of characteristic $p > 0$. If $L/K$ is a finite Galois extension of fields with Galois group $G$, then $L$ is also a complete discrete valuation field with residue field $k$. Here $G$ is of the form $P \rtimes \mathbb{Z}/m$, where

$P$ is a $p$-group and $m$ is prime to $p$. The group $G$ has a filtration $G = G_0 \supseteq G_1 \supseteq \cdots$ for the lower numbering and $G \supseteq G^i$ for $i \in \mathbb{R}_{\geq 0}$ for the upper numbering, with $G^i \supseteq G^j$ when $i \leq j$ (see [Ser79, IV, §1, §3]). The subgroup $G_i$ (resp. $G^i$) is known as the *ith higher ramification group for the lower numbering (resp. the upper numbering)*. One knows that $G_0 = G^0 = G$, $G_1 = P$. For sufficiently small $\epsilon > 0$, $G^\epsilon = P$, and for sufficiently large $i \in \mathbb{Z}$, $G_i = G^i = \{id\}$. Any $i$ such that $G^i \supsetneq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an *upper jump* of the extension $L/K$. Likewise, if $G_i \supsetneq G_{i+1}$, then $i$ is called a *lower jump* of $L/K$. The greatest upper jump (i.e., the greatest $i$ such that $G^i \neq \{id\}$) is called the *conductor* of higher ramification of $L/K$ (for the upper numbering). The upper numbering is invariant under quotients ([Ser79, IV, Proposition 14]). That is, if $H \leq G$ is normal, and $M = L^H$, then the $i$th higher ramification group for the upper numbering for $M/K$ is $G^i/(G^i \cap H)$.

**Lemma 2.8.** *Let $L_1, \ldots, L_\ell$ be Galois extensions of $K$ with compositum $L$ in some algebraic closure of $K$. Denote by $h_i$ the conductor of $L_i/K$ and by $h$ the conductor of $L$. Then $h = \max_i(h_i)$.*

*Proof.* Write $G = \mathrm{Gal}(L/K)$ and $N_i = \mathrm{Gal}(L/L_i)$. Suppose $g \in G^j \subseteq \mathrm{Gal}(L/K)$. Since $L$ is the compositum of the $L_i$, the intersection of the $N_i$ is trivial. So $g$ is trivial iff its image in each $G/N_i$ is trivial. Because the upper numbering is invariant under quotients, this shows that $G^j$ is trivial iff the $j$th higher ramification group for the upper numbering for $L_i/L$ is trivial for all $i$. This means that $h = \max_i(h_i)$. □

**Example 2.9.** Let $K = \mathrm{Frac}(W(k))$, for $k$ algebrically closed of characteristic $p$. Consider the extension $K(\zeta_p, \sqrt[p]{a})/K$, where $a \in K$ is not a $p$th power and satisfies either $v(a) = 1$

or $a = 1 + u$ with $v(u) = 1$. This is a $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$-extension. In the first case, we write $L_1/K$, and we can take a uniformizer of $L_1$ to be $\frac{\zeta_p - 1}{\sqrt[p]{a}}$. In the second case, we write $L_2/K$, and we can take a uniformizer of $L_2$ to be $\frac{\zeta_p - 1}{a - \sqrt[p]{a}}$. A calculation following [Viv04, Theorem 6.3] for $L_1/K$ (or [Viv04, Theorem 5.6] for $L_2/K$) shows that the conductor for the extension $L_1/K$ is $\frac{p}{p-1}$, whereas $L_2/K$ has conductor $\frac{1}{p-1}$.

**Lemma 2.10.** *If $P$ is abelian, then all upper jumps (in particular, the conductor of higher ramification) are in $\frac{1}{m}\mathbb{Z}$.*

*Proof.* Let $L_0 \subset L$ be the fixed field of $L$ under $P$. By the Hasse-Arf theorem ([Ser79, IV, §3]), the upper jumps for the $P$-extension $L/L_0$ are integers. By Herbrand's formula ([Ser79, IV, §3]), the upper jumps for $L/K$ are $\frac{1}{m}$ times those for $L/L_0$. The lemma follows. □

If $P$ is trivial, we say that the extension $L/K$ is *tamely ramified*. Otherwise, we say it is *wildly ramified*. If $A, B$ are the valuation rings of $K, L$, respectively, sometimes we will refer to the conductor or higher ramification groups of the extension $B/A$.

### 2.3.1 Smooth Curves

Let $f : Y \to X$ be a branched cover of smooth, proper, integral curves over $k$. The Hurwitz formula ([Har77, IV §2]) states that

$$2g_Y - 2 = (\deg f)(2g_X - 2) + |R|,$$

where $R$ is the ramification divisor and $|R|$ is its degree. $R$ is supported at the ramification points $y \in Y$. For each ramification point $y \in Y$ with image $x \in X$, we can consider the extension of discrete valuation rings $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,x}$. We say that $y$ is tamely (resp. wildly) ramified when this extension is tamely (resp. wildly) ramified. The degree of the ramification divisor at $y$ can be related to the higher ramification filtrations of $\mathrm{Frac}(\hat{\mathcal{O}}_{Y,y})/\mathrm{Frac}(\hat{\mathcal{O}}_{X,x})$ ([Ser79, IV, Proposition 4]).

In particular, suppose the Galois group $G$ of $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,x}$ is isomorphic to $P \rtimes \mathbb{Z}/m$ with $P$ *cyclic* of order $p^n$. For $1 \leq i \leq n$, write $u_i$ (resp. $j_i$) for the upper (resp. lower) jump such that $G^i$ (resp. $G_i$) is isomorphic to $\mathbb{Z}/p^{n-i+1}$. Write $u_0 = j_0 = 0$. Then $0 = u_0 \leq u_1 < \cdots < u_n$ and $0 = j_0 \leq j_1 < \cdots < j_n$. Let $|R_y|$ be the order of $R$ at $y$.

**Lemma 2.11.** *(i) In terms of the lower jumps, we have*

$$|R_y| = p^n m - 1 + \sum_{i=1}^{n} j_i p^{n-i}(p-1) = p^n m - 1 + \sum_{i=1}^{n} (p^{n-i+1} - 1)(j_i - j_{i-1}).$$

*(ii) In terms of the upper jumps, we have*

$$|R_y| = p^n m - 1 + \sum_{i=1}^{n} m p^{i-1}(p^{n-i+1} - 1)(u_i - u_{i-1}).$$

*Proof.* By [Har77, IV §2], $|R_y|$ is equal to the valuation of the *different* of the extension $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,x}$, where a uniformizer of $\hat{\mathcal{O}}_{Y,y}$ is given valuation 1. By [Ser79, IV, Proposition 4], this different is equal to $\sum_{r=0}^{\infty}(|G_r| - 1)$. Now it is a straightforward exercise to show that (i) holds. Part (ii) follows from part (i) by Herbrand's formula (essentially, the definition of the upper numbering). □

**Remark 2.12.** In the above context, it follows from Herbrand's formula that the conductor $u_n$ is equal to $\frac{1}{m} \sum_{i=1}^{n} \frac{j_i - j_{i-1}}{p^{i-1}}$, which can also be written as $\left(\sum_{i=1}^{n-1} \frac{p-1}{p^i m} j_i\right) + \frac{1}{p^n m} j_n$.

Combining the Hurwitz formula with Lemma 2.11 in the simple case $G \cong \mathbb{Z}/p$, the following corollary is an easy exercise:

**Corollary 2.13.** *Let $f : Y \to \mathbb{P}^1$ be a $\mathbb{Z}/p$-cover of curves over an algebraically closed field $k$ of characteristic $p$, branched at exactly one point of order $p$. If the conductor of higher ramification for the upper numbering at this point is $h$, then the genus of $Y$ is $\frac{(h-1)(p-1)}{2}$.*

We include a well-known lemma (cf. [Pri02, Theorem 1.4.1 (i)]):

**Lemma 2.14.** *Let $k$ be an algebraically closed field of characteristic $p$, and let $f : Y \to \mathbb{P}^1$ be a $\mathbb{Z}/p$-cover, branched only over $\infty$. Then $f$ can be given birationally by an equation $y^p - y = g(x)$, where the terms of $g(x)$ have prime-to-$p$ degree. If $h$ is the conductor of higher ramification at $\infty$, then $h = \deg(g)$.*

### 2.3.2 Non-Smooth Curves

Let $k$ be an algebraically closed field of any characteristic. A *semistable* curve over $k$ is a dimension 1 reduced scheme of finite type over $k$ whose only singularities are ordinary double points. Let $\overline{f} : \overline{Y} \to \overline{X}$ be a finite, flat, separable morphism of projective semistable curves over $k$. Suppose $\overline{X}$ is connected ($\overline{Y}$ need not be connected), and assume that $p_a(\overline{X}) = 0$ (i.e., $\overline{X}$ is a tree of $\mathbb{P}^1$s). Suppose further that $\mathrm{Aut}(\overline{Y}/\overline{X})$ acts transitively on each fiber of $\overline{f}$, that every node of $\overline{Y}$ lies above a node of $\overline{X}$, and that the inertia group of each node of $\overline{Y}$ (on either of the components passing through it) is of order prime to $\mathrm{char}(k)$. Lastly, let $R_{sm}$ be the restriction of the ramification divisor of $\overline{f}$ to the smooth points of $\overline{Y}$. We have the following *generalized Hurwitz formula*

**Proposition 2.15.** $2p_a(\overline{Y}) - 2 = -2(\deg(\overline{f})) + |R_{sm}|$.

*Proof.* Let $C$ be the set of irreducible components of $\overline{X}$ and let $D$ be the set of nodes of $\overline{X}$. Because $\overline{X}$ is a tree of $\mathbb{P}^1$'s, $|C| = |D| + 1$. For each $d \in D$, let $m_d$ be the ramification index of a point $y$ of $\overline{Y}$ above $d$, taken with respect to either irreducible component passing through $\overline{Y}$. This is well-defined because $\mathrm{Aut}(\overline{Y}/\overline{X})$ acts transitively on the fibers. For each $\overline{X}_c \in C$, let $n_c$ be the number of irreducible components of $\overline{Y}$ lying above $\overline{X}_c$. Let $g_c$ be the genus of any of these components (again, well-defined because of the transitive action of $\mathrm{Aut}(\overline{Y}/\overline{X})$). Lastly, let $R_c$ be the ramification divisor on $\overline{Y}_c := \overline{Y} \times_{\overline{X}} \overline{X}_c$ of the map $\overline{Y}_c \to \overline{X}_c$.

Let $S$ be the set of irreducible components and let $T$ be the set of nodes of $\overline{Y}$. By [Liu02, Proposition 10.3.18], $p_a(\overline{Y}) - 1 = |T| + \sum_{\overline{Y}_s \in S}(\mathrm{genus}(\overline{Y}_s) - 1)$. Then we have

$$\begin{aligned}
p_a(\overline{Y}) - 1 &= \sum_{c \in C} n_c(g_c - 1) + \sum_{d \in D} \frac{\deg(\overline{f})}{m_d} \\
&= \sum_{c \in C} \left( n_c \left( -\frac{\deg(\overline{f})}{n_c} \right) + \frac{|R_c|}{2} \right) + \sum_{d \in D} \frac{\deg(\overline{f})}{m_d} \\
&= \frac{|R_{sm}|}{2} - \sum_{c \in C} \deg(\overline{f}) + \sum_{d \in D} \deg(\overline{f}) \left(1 - \frac{1}{m_d}\right) + \sum_{d \in D} \frac{\deg(\overline{f})}{m_d} \\
&= \frac{|R_{sm}|}{2} + (|D| - |C|)(\deg(\overline{f})) \\
&= \frac{|R_{sm}|}{2} - \deg(\overline{f}),
\end{aligned}$$

with the second equality coming from the Hurwitz formula on each component and the third equality coming from separating out those parts of each $R_c$ included in $R_{sm}$. The formula follows by multiplying both sides by 2. $\square$

We maintain the assumptions of Proposition 2.15. Furthermore, let $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with $p \nmid m$, where $\mathrm{char}(k)$ is either $p$ or 0, and suppose that $G \leq \mathrm{Aut}(\overline{Y}/\overline{X})$ acts generically freely on $\overline{Y}$ and acts transitively on all fibers of $\overline{f}$. In particular, $\deg(f) = p^n m$. Let $H \leq G$ be the unique subgroup of order $p$. Write $B$ for the set of smooth points of $\overline{X}$ with branching index prime to $\text{char}(k)$, and $B'$ for the set of smooth branch points of $x$ where $\text{char}(k)$ divides the branching index. If $x_b \in B$, write $m_b$ for the branching index at $x_b$, and if $x_b \in B'$, write $\sigma_b$ for the conductor of higher ramification for a point of $\overline{Y}$ above $x_b$.

**Corollary 2.16.** *The difference $p_a(\overline{Y}) - p_a(\overline{Y}/H)$ is equal to*

$$\frac{1}{2}p^{n-1}(p-1)m\left(\sum_{b\in B}\left(1-\frac{1}{m_b}\right)+\sum_{b\in B'}(\sigma_b+1)-2\right).$$

*Proof.* Let $R'_{sm}$ be the restriction of the ramification divisor of $\overline{Y}/H \to \overline{X}$ to the smooth locus of $\overline{Y}/H$. By Proposition 2.15, we have

$$p_a(\overline{Y}) - p_a(\overline{Y}/H) = -p^{n-1}(p-1)m + \frac{|R_{sm}| - |R'_{sm}|}{2}. \tag{2.3.1}$$

For each $x_b \in B$, the contribution of $x_b$ to $|R_{sm}|$ is $p^n m(1-\frac{1}{m_b})$, and to $|R'_{sm}|$ it is $p^{n-1}m(1-\frac{1}{m_b})$. For each $x_b \in B'$, suppose the ramification index above $x_b$ is equal to $p^{n'}m'$, with $p \nmid m'$. Then the contribution of $b$ to $|R_{sm}|$ is

$$p^{n-n'}\frac{m}{m'}\left(p^{n'}m' - 1 + \sum_{i=1}^{n'} m'p^{i-1}(p^{n'-i+1}-1)(u_i - u_{i-1})\right),$$

where the $u_i$ are the upper jumps for a point $y_b$ above $x_b$ (Lemma 2.11). Similarly, the contribution of $b$ to $|R'_{sm}|$ is

$$p^{n-n'}\frac{m}{m'}\left(p^{n'-1}m' - 1 + \sum_{i=1}^{n'-1} m'p^{i-1}(p^{n'-i}-1)(u_i - u_{i-1})\right).$$

Since, by definition, $\sigma_b = u_n$, a straightforward calculation shows that the difference in these contributions is $p^{n-1}(p-1)m(\sigma_b+1)$. So

$$|R_{sm}| - |R'_{sm}| = p^{n-1}(p-1)m\left(\sum_{b\in B}\left(1-\frac{1}{m_b}\right) + \sum_{b\in B'}(\sigma_b+1)\right).$$

Combining this with (2.3.1) yields the corollary. □

## 2.4 Semistable models of $\mathbb{P}^1$

We now introduce some notation that will be used for the remainder of Chapter 2. Let $X \cong \mathbb{P}^1_K$, where $K$ is a characteristic zero complete discretely valued field with algebraically closed residue field $k$ of characteristic $p > 0$ (e.g., $K = \mathbb{Q}_p^{ur}$). Let $R$ be the valuation ring of $K$. Write $v$ for the valuation on $R$. We normalize by setting $v(p) = 1$.

### 2.4.1 Models

Fix a smooth model $X_R$ of $X$ over $R$, i.e., a smooth, flat, proper $R$-scheme $X_R$ such that $X_R \otimes_R K \cong X$. Then there is an element $T \in K(X)$ such that $K(T) \cong K(X)$ and the local ring at the generic point of the special fiber of $X_R$ is the valuation ring of $K(T)$ corresponding to the Gauss valuation. We say that our model corresponds to the Gauss valuation on $K(T)$, and we call $T$ a *coordinate* of $X_R$. Conversely, if $T$ is any rational function on $X$ such that $K(T) \cong K(X)$, there is a smooth model $X_R$ of $X$ such that $T$ is a coordinate of $X_R$. Since the automorphism group of $K(T)/K$ is $PGL_2(K)$, whereas the automorphism group of $K(T)_{\text{gauss}}/K$ as an extension of valued fields is $PGL_2(R)$, this shows that smooth models of $X$ up to automorphism are in bijection with $PGL_2(K)/PGL_2(R)$. This bijection becomes canonical once we pick a preferred model $X_R$.

Let $X_R$ be a semistable model of $X$ over $R$ (i.e., its fibers are semistable, see §2.3.2) The special fiber of such a model $X_R$ is a tree-like configuration of $\mathbb{P}^1_k$'s. Each irreducible component of the special fiber $\overline{X}$ of $X_R$ corresponds to a smooth model of $X$, and thus a valuation ring of $K(X)$. Such models will be used in §2.5, and throughout the thesis.

### 2.4.2 Disks and Annuli

We give a brief overview here. For more details, see [Hen98].

Let $X_R$ be a semistable model for $X = \mathbb{P}^1_K$. Suppose $x$ is a smooth point of the special fiber of $X_R$ on the irreducible component $\overline{W}$. Let $T$ be a coordinate of the smooth model of $X$ with special fiber $\overline{W}$ such that $T = 0$ specializes to $x$. Then the set of points of $X$ which specialize to $x$ is the *open $p$-adic disk* given by $v(T) > 0$. The complete local ring of $x$ in $X_R$ is isomorphic to $R[[T]]$. For our purposes, a general open (resp. closed) $p$-adic disk is a subset $D \subset \mathbb{P}^1(K)$ such that there is a rational function $T$ with $K(T) \cong K(\mathbb{P}^1)$ and $D$ is given by $v(T) > 0$ (resp. $v(T) \geq 0$).

Now, let $x$ be a nodal point of the special fiber of $X_R$, at the intersection of components $\overline{W}$ and $\overline{W}'$. Then the set of points of $X$ which specialize to $x$ is an *open annulus*. If $T$ is a coordinate on the smooth model of $X$ with special fiber $\overline{W}$ such that $T = 0$ does *not* specialize to $x$, then the annulus is given by $0 < v(T) < e$ for some $e \in v(K^\times)$. The complete local ring of $x$ in $X_R$ is isomorphic to $R[[T,U]]/(TU - p^e)$. For our purposes, a general open annulus of *épaisseur* $e$ is a subset $A \subset \mathbb{P}^1(K)$ such that there is a rational function $T$ with $K(T) \cong K(\mathbb{P}^1)$ and $A$ is given by $0 < v(T) < e$. Observe that $e$ is independent of the coordinate.

Suppose we have a preferred coordinate $T$ on $X$ and a semistable model $X_R$ of $X$ whose special fiber $\overline{X}$ contains an irreducible component $\overline{X}_0$ corresponding to the coordinate $T$. If $\overline{W}$ is any irreducible component of $\overline{X}$ other than $\overline{X}_0$, then since $\overline{X}$ is a tree of $\mathbb{P}^1$'s, there is a unique nonrepeating sequence of consecutive, intersecting components $\overline{W}, \ldots, \overline{X}_0$. Let $\overline{W}'$ be the component in this sequence that intersects $\overline{W}$. Then the set of points in $X(K)$ that specialize to the connected component of $\overline{W}$ in $\overline{X}\backslash\overline{W}'$ is a closed $p$-adic disk $D$. If the established preferred coordinate (equivalently, the preferred component $\overline{X}_0$) is clear, we will abuse language and refer to the component $\overline{W}$ as *corresponding to the disk* $D$, and vice versa.

## 2.5 Stable reduction

We continue the notations of §2.4, but we allow $X$ to be a smooth, proper, geometrically integral curve of any genus $g_X$, so long as $X$ has a smooth model $X_R$ over $R$. Let $f : Y \to X$ be a $G$-Galois cover defined over $K$, with $G$ any finite group, such that the branch points of $f$ are defined over $K$ and their specializations do not collide on the special fiber of $X_R$. Assume that $2g_X - 2 + r \geq 1$, where $r$ is the number of branch points of $f$. By a theorem of Deligne and Mumford ([DM69, Corollary 2.7]), combined with work of Raynaud ([Ray90], [Ray99]), there is a minimal finite extension $K^{st}/K$ with ring of integers $R^{st}$, and a unique semistable model $X^{st}$ of $X_{K^{st}} = X \otimes_K K^{st}$, such that $X^{st}$ is a blowup of $X_{R^{st}} = X_R \otimes_R R^{st}$ centered at closed points of the special fiber, $X^{st}$ is normal, and the normalization $Y^{st}$ of $X^{st}$ in $K^{st}(Y)$ has the following properties:

- The special fiber $\overline{Y}$ of $Y^{st}$ is semistable, i.e., it is reduced, and has only nodes for singularities. Furthermore, the nodes of $\overline{Y}$ lie above nodes of the special fiber $\overline{X}$ of $X^{st}$.

- The ramification points of $f_{K^{st}} = f \otimes_K K^{st}$ specialize to *distinct* smooth points of $\overline{Y}$.

- Any genus zero irreducible component of $\overline{Y}$ contains at least three marked points (i.e., ramification points or points of intersection with the rest of $\overline{Y}$).

The map $f^{st} : Y^{st} \to X^{st}$ is called the *stable model* of $f$ and the field $K^{st}$ is called the minimal field of definition of the stable model of $f$. If we are working over a finite field extension $K'/K^{st}$ with ring of integers $R'$, we will sometimes abuse language and call $f^{st} \otimes_{R^{st}} R'$ the stable model of $f$. For each $\sigma \in G_K$, $\sigma$ acts on $\overline{Y}$ and this action commutes with $G$. Then it is known ([Ray99, Théorème 2.2.2]) that the extension $K^{st}/K$ is the fixed field of the group $\Gamma^{st} \leq G_K$ consisting of those $\sigma \in G_K$ such that $\sigma$ acts trivially on $\overline{Y}$. Thus $K^{st}$ is clearly Galois over $K$.

If $\overline{Y}$ is smooth, the cover $f : Y \to X$ is said to have *potentially good reduction.* If $\overline{Y}$ can be contracted to a smooth curve by blowing down components of genus zero, then the curve $Y$ is said to have potentially good reduction. If $f$ or $Y$ does not have potentially good reduction, it is said to have *bad reduction.* In any case, the special fiber $\overline{f} : \overline{Y} \to \overline{X}$ of the stable model is called the *stable reduction* of $f$. The action of $G$ on $Y$ extends to the stable reduction $\overline{Y}$ and $\overline{Y}/G \cong \overline{X}$. The strict transform of the special fiber of

$X_{R^{st}}$ in $\overline{X}$ is called the *original component*, and will be denoted $\overline{X}_0$. We orient the tree of $\overline{X}$ outward from the original component $\overline{X}_0$. This induces a partial order $\prec$ where irreducible components $\overline{W}$ and $\overline{W}'$ of $\overline{X}$ satisfy $\overline{W} \prec \overline{W}'$ if and only if $\overline{W}'$ lies in the connected component of $\overline{X} \backslash \overline{W}$ not containing $\overline{X}_0$. If $w$ is a point of $\overline{X}$, we will write $w \prec \overline{W}'$ if there is a component $\overline{W} \ni w$ such that $\overline{W} \prec \overline{W}'$. Also, we write $w \succ \overline{W}$ if there is a component $\overline{W}' \ni w$ such that $\overline{W}' \succ \overline{W}$. Lastly, if $w$ and $w'$ are points of $\overline{X}$, we write $w \prec w'$ if $w \neq w'$ and if there exist components $\overline{W}$ and $\overline{W}'$ containing $w$ and $w'$ respectively such that $\overline{W} \prec \overline{W}'$. The symbol $\preceq$ has the obvious meaning in the case of two components or two points. A maximal component for $\prec$ is called a *tail*. All other components are called *interior components*.

### 2.5.1 Inertia Groups of the Stable Reduction

**Proposition 2.17.** *The inertia groups of $\overline{f} : \overline{Y} \to \overline{X}$ at points of $\overline{Y}$ are as follows (note that points in the same $G$-orbit have conjugate inertia groups):*

*(i) At the generic points of irreducible components, the inertia groups are $p$-groups.*

*(ii) At each node, the inertia group is an extension of a cyclic, prime-to-$p$ order group by a $p$-group generated by the inertia groups of the generic points of the crossing components.*

*(iii) If a point $b_i \in Y$ above a branch point $a_i \in X$ specializes to a smooth point $\overline{b}_i$ on a component $\overline{V}$ of $\overline{Y}$, then the inertia group at $\overline{b}_i$ is an extension of the prime-to-$p$ part of the inertia group at $b_i$ by the inertia group of the generic point of $\overline{V}$.*

*(iv) At all other points $q$ (automatically smooth, closed), the inertia group is equal to the inertia group of the generic point of the irreducible component of $\overline{Y}$ containing $q$.*

*Proof.* [Ray99, Proposition 2.4.11]. □

For the rest of this subsection, assume $G$ has a *cyclic* $p$-Sylow subgroup. When $G$ has a cyclic $p$-Sylow subgroup, the inertia groups above a generic point of an irreducible component $\overline{W} \subset \overline{X}$ are conjugate cyclic groups of $p$-power order. If they are of order $p^i$, we call $\overline{W}$ a *$p^i$-component*. If $i = 0$, we call $\overline{W}$ an *étale component*, and if $i > 0$, we call $\overline{W}$ an *inseparable* component. This nomenclature comes from the fact that for an inseparable component $\overline{W}$, $Y \times_X \overline{W} \to \overline{W}$ corresponds to an inseparable extension of the function field $k(\overline{W})$.

**Corollary 2.18.** *If $\overline{V}$ and $\overline{V}'$ are two adjacent irreducible components of $\overline{Y}$, and if $I_V$ and $I_{V'}$ are the inertia groups of their generic points, then either $I_V \subseteq I_{V'}$ or vice versa.*

*Proof.* Let $q$ be a point of intersection of $\overline{V}$ and $\overline{V}'$ and let $I_q$ be its inertia group. Then the $p$-part of the $I_q$ is a cyclic $p$-group, generated by the two cyclic $p$-groups $I_V$ and $I_{V'}$. Since the subgroups of a cyclic $p$-group are totally ordered, the corollary follows. □

**Proposition 2.19.** *If $x \in X$ is branched of index $p^a s$, where $p \nmid s$, then $x$ specializes to a $p^a$-component.*

*Proof.* By Proposition 2.17 (iii) and the definition of the stable model, $x$ specializes to a smooth point of a component whose generic inertia has order at least $p^a$. Because our definition of the stable model requires the specializations of the $|G|/p^a s$ ramification points above $a$ to be disjoint, the specialization of $x$ must have a fiber with cardinality a multiple of $|G|/p^a s$. This shows that $x$ must specialize to a component with inertia at most $p^a$. $\square$

**Definition 2.20.** Let $\overline{W}$ be an irreducible component of $\overline{X}$. We call the stable reduction $\overline{f}$ of $f$ *monotonic from* $\overline{W}$ if for every $\overline{W} \preceq \overline{W}' \preceq \overline{W}''$ such that $\overline{W}'$ is a $p^i$-component and $\overline{W}''$ is a $p^j$-component, we have $i \geq j$. In other words, the stable reduction is monotonic from $\overline{W}$ if the generic inertia does not increase as we move outward from $\overline{W}$ along $\overline{X}$. If $\overline{f}$ is monotonic from the original component $\overline{X}_0$, we say simply that $\overline{f}$ is *monotonic*.

**Remark 2.21.** Let $P$ be a $p$-Sylow subgroup of $G$. We will eventually show (Proposition 3.10) that if $f$ is a three-point $G$-cover such that $G$ is $p$-solvable or $m_G = 2$, then $f$ will *always* have monotonic stable reduction. So for the types of covers considered in Theorem 1.4, we have monotonic stable reduction.

**Proposition 2.22** ([Ray99], Proposition 2.4.8)**.** *If $\overline{W}$ is an étale component of $\overline{X}$, then $\overline{W}$ is a tail.*

**Proposition 2.23** ([Ray99], Corollaire 2.4.10)**.** *If $\overline{X} = \overline{X}_0$ and the genus of $Y$ is at least 2, then $Y_R$ is smooth, and $\overline{Y}$ is generically étale over $\overline{X}$.*

**Proposition 2.24.** *Assume $\overline{X}$ is not smooth. If $\overline{W}$ is a tail of $\overline{X}$ which is a $p^a$-component, then the component $\overline{W}'$ that intersects $\overline{W}$ is a $p^b$-component with $b > a$.*

*Proof.* Since $\overline{X}$ is not smooth, $\overline{W}$ is not the original component. Assume that the proposition is false. The proof is virtually the same as the proof of [Ray99, Lemme 3.1.2]. Let

$\overline{V}$ be an irreducible component of $\overline{Y}$ lying above the genus zero component $\overline{W}$, and let $I \cong \mathbb{Z}/p^a$ be the inertia group of $\overline{V}$. Consider $\overline{Y}/I =: \overline{Y}^{\wedge}$. The image $\overline{V}^{\wedge}$ of $\overline{V}$ in $\overline{Y}^{\wedge}$ is now generically étale and tamely ramified over $\overline{W}$. The only possible branch points are the point of intersection of $\overline{W}$ and $\overline{W}'$, and the specialization of at most one point $a_i$ to $\overline{W}$. Since there are at most two branch points, and they are tame, $\overline{V}^{\wedge}$ is totally ramified at these points, and thus it has genus zero and is connected to $\overline{Y}^{\wedge}$ at only one point. But then the same is true for $\overline{V}$ and $\overline{Y}$, as quotienting out by $I$ corresponds to a radicial extension, which does not change the genus. This contradicts the definition of the stable model, as $\overline{V}$ has insufficiently many marked points. □

Note that Proposition 2.24 shows that if $p$ exactly divides $|G|$, then there are no inseparable tails. But there can be inseparable tails if a higher power of $p$ divides $|G|$. An étale tail of $\overline{X}$ is called *primitive* if it contains a branch point other than the point at which it intersects the rest of $\overline{X}$. Otherwise it is called *new*. This follows [Ray99]. An inseparable tail that does not contain the specialization of any branch point will be called a *new inseparable tail*.

**Definition 2.25.** Consider an étale tail $\overline{X}_b$ of $\overline{X}$. Suppose $\overline{X}_b$ intersects the rest of $\overline{X}$ at $x_b$. Let $\overline{Y}_b$ be a component of $\overline{Y}$ lying above $\overline{X}_b$, and let $y_b$ be a point lying above $x_b$. The *generalized ramification invariant* $\sigma_b$ for $\overline{X}_b$ is the conductor of higher ramification for the extension $\hat{\mathcal{O}}_{\overline{Y}_b,y_b}/\hat{\mathcal{O}}_{\overline{X}_b,x_b}$ with respect to the upper numbering (see §2.3).

**Lemma 2.26.** *The generalized ramification invariants $\sigma_b$ lie in $\frac{1}{m_G}\mathbb{Z}$.*

*Proof.* The extension $\hat{\mathcal{O}}_{\overline{Y}_b,y_b}/\hat{\mathcal{O}}_{\overline{X}_b,x_b}$ has Galois group $I_{y_b}$ of the form $\mathbb{Z}/p^r \rtimes \mathbb{Z}/\ell$ for some

$r, \ell$. Since $I_{y_b} \subseteq G$, the quotient of $I_{y_b}$ by its maximal prime-to-$p$ central subgroup is $\mathbb{Z}/p^r \rtimes \mathbb{Z}/m'$, where $m' | m_G$. The ramification invariant over $x$ for $\overline{Y} \to \overline{X}_b$ is the same as for $\overline{Y}/H \to \overline{X}_b$, see Remark 3.3. So $\sigma_b \in \frac{1}{m'}\mathbb{Z} \subseteq \frac{1}{m_G}\mathbb{Z}$. □

## 2.6 The auxiliary cover

Retain the assumptions from the beginning of §2.5 (in particular, $G$ need not have a cyclic $p$-Sylow subgroup). Assume that $f : Y \to X$ is a $G$-cover defined over $K$ as in §2.5 with bad reduction, so that $\overline{X}$ is not just the original component. Following [Ray99, §3.2], we can construct an *auxiliary cover* $f : Y^{aux} \to X$ with (modified) stable model $(f^{aux})^{st} : (Y^{aux})^{st} \to X^{st}$ and (modified) stable reduction $\overline{f}^{aux} : \overline{Y}^{aux} \to \overline{X}$, defined over some finite extension $R'$ of $R$, with the following properties:

- Above an étale neighborhood of the union of those components of $\overline{X}$ other than étale tails, the cover $Y^{st} \to X^{st}$ is induced (see §1.4) from the connected Galois cover $(Y^{aux})^{st} \to X^{st}$.

- The modified stable reduction $\overline{Y}^{aux} \to \overline{X}$ of $Y^{aux} \to X$ is given first by replacing the components of $\overline{Y}$ above étale tails with Katz-Gabber covers (see [Kat86, Theorem 1.4.1]), and then taking one connected component.

We will explain what "modified" means in a remark following the construction of the auxiliary cover. The construction is almost entirely the same as in [Ray99, §3.2], and we will not repeat the details. Instead, we give an overview, and we mention where our construction differs from Raynaud's.

Let $B_{\text{ét}}$ index the étale tails of $\overline{X}$. Subdividing $B_{\text{ét}}$, we index the set of primitive tails by $B_{\text{prim}}$ and the set of new tails by $B_{\text{new}}$. We will write $\overline{X}_b$ for the tail indexed by some $b \in B_{\text{ét}}$.

The construction proceeds as follows: From $\overline{Y}$ remove all of the components that lie above the *étale* tails of $\overline{X}$ (as opposed to *all* the tails—this is the only thing that needs to be done differently than in [Ray99], where all tails were étale). Now, what remains of $\overline{Y}$ is possibly disconnected. We throw out all but one connected component, and call what remains $\overline{V}$. For each $\overline{X}_b$, $b \in B_{\text{prim}}$, let $a_b$ be the branch point of $f$ specializing to $\overline{X}_b$, let $x_b$ be the point where $\overline{X}_b$ intersects the rest of $\overline{X}$, and let $p^r m_b$ be the index of ramification above $x_b$, with $m_b$ prime-to-$p$. At each point of $\overline{V}$ above $x_b$, we attach to $\overline{V}$ a *Katz-Gabber* cover (cf. [Kat86, Theorem 1.4.1], [Ray99, Théorème 3.2.1]), branched of order $m_b$ (with inertia groups isomorphic to $\mathbb{Z}/m_b$) at the specialization of $a_b$ and of order $p^r m_b$ (with inetria groups isomorphic to $\mathbb{Z}/p^r \rtimes \mathbb{Z}/m_b$) at $x_b$, where $p^r$ is the order of inertia of the generic point of the interior component containing $x_b$. We choose our Katz-Gabber cover so that above the complete local ring of $x_b$ on $\overline{X}_b$, it is isomorphic to the original cover. It is the composition of a cyclic cover of order $m_b$ with a cover of order $p^r$ branched at one point. Note that if $m_b = 1$, we have eliminated a branch point of the original cover.

For each $b \in B_{\text{new}}$, we do the same thing, except that we *introduce* an (arbitrary) branch point $\alpha_b$ of ramification index $m_b$ somewhere on the new tail under consideration. Let $\overline{Y}^{aux} \to \overline{X}$ be the cover of $k$-schemes we have just constructed.

As in [Ray99, §3], one shows that we can lift $\overline{Y}^{aux} \to \overline{X}$ to a cover $(f^{aux})^{st} : (Y^{aux})^{st} \to X^{st}$ over $R'$. The generic fiber $f^{aux} : Y^{aux} \to X$ is branched exactly at the branch points of $f$ and at a new point $a_b$ of index $m_b$ for each new tail $b \in B_{\text{new}}$ (unless one of the branch points of $f$ was eliminated, as noted above). Each $a_b$ specializes to the corresponding branch point $\alpha_b$ introduced in the previous paragraph. Keep in mind that there is some choice here in how to pick the new branch points—depending on the choice of $\alpha_b$, we can end up choosing $a_b$ to be *any* point that specializes to $\overline{X}_b$. The generic fiber $f^{aux}$ is referred to as the *auxiliary cover*. It is a Galois cover with Galois group $G^{aux}$ (equal to the decomposition group of $\overline{V}$ inside $G$). The group $G^{aux}$ is a subgroup of $G$. It satisfies the properties that we have mentioned for the auxiliary cover.

**Remark 2.27.** It may happen that the actual stable reduction of the cover $f^{aux} : Y^{aux} \to X$ is a contraction of $\overline{Y}^{aux}$ as defined above (or that it is not even defined, as we may have eliminated a branch point by passing to the auxiliary cover). This happens only if $\overline{X}$ has a primitive tail $\overline{X}_b$ for which $m_b = 1$, and for which the Katz-Gabber cover inserted above $\overline{X}_b$ has genus zero. Then this tail, and possibly some components inward, would be contracted in the stable reduction of $f^{aux}$. We use the term *modified* stable reduction to mean that we do not perform this contraction, so $\overline{f} : \overline{Y}^{aux} \to \overline{X}$ is indeed as given in the construction above. When we are not in this situation, the modified stable model is the same as the stable model.

If we are interested in understanding fields of moduli (or more generally, fields of definition of the stable model), it is in some sense good enough to understand the auxiliary cover, as the following lemma shows.

**Lemma 2.28.** *If $K^{st}$ is the smallest field over which the modified stable model $(f^{aux})^{st}$ : $(Y^{aux})^{st} \to X^{st}$ of the auxiliary cover $f^{aux}$ is defined, then the stable model $f^{st}$ of $f$ is defined over $K^{st}$.*

*Proof.* (cf. [Wew03b], Theorem 4.5) Take $\sigma \in \Gamma^{st}$, the absolute Galois group of $K^{st}$. We must show that $f^{\sigma} \cong f$ and that $\sigma$ acts trivially on the stable reduction $\overline{f} : \overline{Y} \to \overline{X}$ of $f$. Let $\hat{f} : \hat{Y} \to \hat{X}$ be the formal completion of $f^{st}$ at the special fiber and let $\hat{f}^{aux} : \hat{Y}^{aux} \to \hat{X}$ be the formal completion of $(f^{aux})^{st}$ at the special fiber. For each étale tail $\overline{X}_b$ of $\overline{X}$, let $x_b$ be the intersection of $\overline{X}_b$ with the rest of $\overline{X}$. Write $\mathcal{D}_b$ for the formal completion of $\overline{X}_b \backslash \{x_b\}$ in $X_{R^{st}}$. $\mathcal{D}_b$ is a closed formal disk, which is certainly preserved by $\sigma$. Also, let $\mathcal{U}$ be the formal completion of $\overline{X} \setminus \bigcup_b \overline{X}_b$ in $X_{R^{st}}$.

Write $\mathcal{V} = \hat{Y} \times_{\hat{X}} \mathcal{U}$. We know from the construction of the auxiliary cover that

$$\mathcal{V} = \mathrm{Ind}_{G^{aux}}^{G} \; \hat{Y}^{aux} \times_{\hat{X}} \mathcal{U}.$$

Since $\sigma$ preserves the auxiliary cover and acts trivially on its special fiber, $\sigma$ acts as an automorphism on $\mathcal{V}$ and acts trivially on its special fiber. By uniqueness of tame lifting, $\mathcal{E}_b := \hat{Y} \times_{\hat{X}} \mathcal{D}_b$ is the unique lift of $\overline{Y} \times_{\overline{X}} (\overline{X}_b \backslash \{x_b\})$ to a cover of $\mathcal{D}_b$ (if $\overline{X}_b$ is primitive, we require the lifting to fix the branch point). This means that $\sigma$ acts as an automorphism on $\hat{Y} \times_{\hat{X}} \mathcal{D}_b$ as well.

Define $\mathcal{B}_b := \mathcal{U} \times_{\hat{X}} \mathcal{D}_b$, the boundary of the disk $\mathcal{D}_b$. A $G$-cover of formal schemes $\hat{Y} \to \hat{X}$ such that $\hat{Y} \times_{\hat{X}} \mathcal{U} \cong \mathcal{V}$ and $\hat{Y} \times_{\hat{X}} \mathcal{D}_b \cong \mathcal{E}_b$ is determined by a patching isomorphism

$$\varphi_b : \mathcal{V} \times_{\mathcal{U}} \mathcal{B}_b \xrightarrow{\sim} \mathcal{E}_b \times_{\mathcal{D}_b} \mathcal{B}_b$$

for each $b$. Then the isomorphism $\varphi_b$ is determined by its restriction $\overline{\varphi}_b$ to the special fiber.

Let $\overline{X}_{b,\infty}$ be the generic point of the completion of $\overline{X}_b$ at $x_b$, and define $\overline{Y}_{b,\infty}$ (resp. $\overline{Y}^{aux}_{b,\infty}$) $:= \overline{Y} \times_{\overline{X}} \overline{X}^{aux}_{b,\infty}$ (resp. $\overline{Y} \times_{\overline{X}} \overline{X}_{b,\infty}$). Then $\overline{Y}_{b,\infty} = \mathrm{Ind}^G_{G^{aux}} \overline{Y}^{aux}_{b,\infty}$. Since $\sigma$ acts trivially on $\overline{Y}^{aux}_{b,\infty}$, it acts trivially on $\overline{Y}_{b,\infty}$, and thus on the isomorphism $\varphi_b$. Thus $\hat{f}^\sigma \cong \hat{f}$, and by Grothendieck's Existence Theorem, $f^\sigma \cong f$.

Lastly, we must check that $\sigma$ acts trivially on $\overline{f}$. This is clear away from the étale tails. Now, for each étale tail $\overline{X}_b$, we know $\sigma$ acts trivially on $\overline{X}_b$, so it must act vertically on $\overline{Y}_b := \overline{Y} \times_{\overline{X}} \overline{X}_b$. But $\sigma$ also acts trivially on $\overline{Y}^{aux}_{\infty,b}$. Since $\overline{Y}_{\infty,b}$ is induced from $\overline{Y}^{aux}_{\infty,b}$, $\sigma$ acts trivially on $\overline{Y}_{\infty,b}$. Therefore, $\sigma$ acts trivially on $\overline{Y}_b$. □

For us, the most important thing about the auxiliary cover is the following:

**Proposition 2.29.** *If we assume that a $p$-Sylow subgroup of $G$ is* cyclic*, then the group $G^{aux}$ has a normal subgroup of order $p$.*

*Proof.* Let $\overline{U}$ be an irreducible component of $\overline{Y}^{aux}$ lying above the original component $\overline{X}_0$. Since we are assuming that $\overline{X}$ is not smooth, $\overline{X}_0$ is not a tail and thus any component of $\overline{Y}^{aux}$ lying above $\overline{X}_0$ is generically inseparable by Proposition 2.22. So let $I_U$ be the inertia group of the generic point of $\overline{U}$ and let $Q_1$ be the unique subgroup of $I_U$ of order $p$. We claim $Q_1$ is normal in $G^{aux}$. To see this, note that conjugation by an element in $G^{aux}$ will send $I_U$ to the inertia group $I_{U'}$ of the generic point of some other component $\overline{U}'$ of $\overline{Y}^{aux}$ lying above $\overline{X}_0$. But by the construction of the auxiliary cover, $\overline{U}'$ can be connected to $\overline{U}$ by a path that passes only through components that do not lie above étale tails. The inertia groups of these components are all nontrivial cyclic $p$-groups, by Proposition 2.22.

We know from Corollary 2.18 that for any two such adjacent components, the inertia group of one contains the inertia group of the other. Since a nontrivial cyclic $p$-group contains exactly one subgroup of order $p$, both inertia groups contain the *same* subgroup of order $p$. Thus $I_{U'}$ contains $Q_1$, and we are done. □

Lastly, in the case that a $p$-Sylow subgroup of $G$ is cyclic, we make a further simplification of the auxiliary cover, as in [Ray99, Remarque 3.1.8]. By Proposition 2.29 and Corollary 2.5, $G^{aux}$ is $p$-solvable. Its quotient by its maximal normal prime-to-$p$ subgroup $N$ is, by Corollary 2.4, isomorphic to $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, where $m|(p-1)$ and the action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful. So we replace $Y^{aux}$ with $Y^{str} := Y^{aux}/N$, which is a branched $G^{str} := G^{aux}/N$-cover of $X$. This is called the *strong auxiliary cover*. Constructing the strong auxiliary cover is one of the key places where it is essential to assume that a $p$-Sylow subgroup of $G$ is cyclic, as otherwise the $G^{aux}$ does not necessarily have such nice group-theoretical properties.

The branching on the generic fiber of the strong auxiliary cover is as follows: At each point of $X$ where the branching index of $f$ was divisible by $p$, the branching index of $f^{str}$ is a power of $p$ (as $G^{str}$ has only elements of $p$-power order and of prime-to-$p$ order). At each branch point specializing to an étale tail $b \in B_{\text{ét}}$, the ramification index is $m_b$, where $m_b|(p-1)$ (cf. [Ray99, §3.3.2]).

The following two lemmas show that it will generally suffice to look at the strong auxiliary cover instead of the auxiliary cover. Let $((f^{str})^{st})' : ((Y^{str})^{st})' \to (X^{st})'$ be the minimal modification of $(f^{str})^{st}$ centered on the special fiber such that all images of ramification points of $f^{aux} : Y^{aux} \to X$ in $Y^{str}$ specialize to distinct points on the special fiber. This modification is only necessary if there exists such a point on $Y^{str}$ that is not a ramification point of $f^{str}$. Write $(\overline{f}^{str})' : (\overline{Y}^{str})' \to \overline{X}'$ for the stable reduction.

**Lemma 2.30.** *If $L$ is a field of definition of $((f^{str})^{st})'$, then the stable model of $q : Y^{aux} \to Y^{str}$ can be defined over an extension $M$ of $L$ such that $p \nmid [M : L]$.*

*Proof.* We first claim that all branch points of $q$ are rational over $L$. Let $\sigma \in G_L$. Then $\sigma$ permutes the points above any given branch point of $f^{aux}$. By the construction of $((f^{str})^{st})'$, each of these points specializes to a different point on $(\overline{Y}^{str})'$. Since $\sigma$ acts trivially on $(\overline{Y}^{str})'$, we have that $\sigma$ fixes each branch point of $q$. Thus, the ramification points are $L$-rational. In particular, the branch points of $q$ are $L$-rational and tame.

Now we apply a variant of [Saï97, Théorème 3.7] to conclude. In [Saï97], it is assumed that $\overline{Y}^{aux} \to \overline{Y}^{str}$ has no smooth branch points. But the argument works so long as we have a canonical $L$-rational point specializing to each of these smooth branch points, because we can use [Ful69, Theorem 4.10] to lift the disk covers corresponding to these branch points. See [Saï97] for more details. □

**Lemma 2.31.** *Let $f : Z \to X$ be a $G$-Galois cover with bad reduction as in §2.5 (we do not assume that a $p$-Sylow subgroup of $G$ is cyclic). Suppose $G$ has a normal subgroup $N$ such that $p \nmid |N|$. Let $f' : Z' := Z/N \to X$ be the quotient cover. Let $M$ be a field over which the stable model of both $f' : Z' \to X$ and the quotient map $q : Z \to Z'$ are defined. Then the stable model of $f$ is defined over an extension $M'$ of $M$ such that $p \nmid [M' : M]$.*

*Proof.* We first show that the field of moduli of $f$ is defined over a prime-to-$p$ extension of $M$. Let $\sigma \in G_M$. Clearly, $f$ is defined as a *mere* cover over $M$. From the fundamental exact sequence (cf. §1.1), we obtain a homomorphism $h : G_M \to \mathrm{Out}(G)$. By our assumptions on $f'$ and $q'$, the image of $h$ consists of those $\overline{\alpha} \in \mathrm{Out}(G)$ which restrict to an inner automorphism on $N$ and descend to an inner automorphism on $G/N$. Take a representative $\alpha$ for $\overline{\alpha}$ in $\mathrm{Aut}(G)$ that fixes $N$ and $G/N$ pointwise. If $g \in G$, then $\alpha(g) = sg$, for some $s \in N$. By our assumptions on $\alpha$, $\alpha^i(g) = s^i g$. Since $s \in N$, we know $s^{|N|}$ is trivial, so $\alpha^{|N|}$ is trivial. So $G_M/(\ker h)$ has prime-to-$p$ order, and thus the field of moduli $M''$ of $f$ (relative to $M$) is a prime-to-$p$ extension of $M$.

Now, let $M'/M''$ be the minimal extension over which the stable model of $f$ is defined. Then $\mathrm{Gal}(M'/M'')$ acts on $\overline{f}$. By assumption, this action descends to a trivial action on $\overline{f'}$, and is trivial on $q$. Thus it is trivial on $\overline{f}$. So in fact $M' = M''$, which satisfies $p \nmid [M'' : M]$. □

In particular, Lemma 2.31 applies when $Z = Y^{aux}$ and $Z' = Y^{str}$.

While the Galois group of the (strong) auxiliary cover is simpler than the original Galois group of $f$, we generally are made to pay for this with the introduction of new branch points. Understanding where these branch points appear is key to understanding the minimal field of definition of the stable reduction of the auxiliary cover. See in particular §4.3.

## 2.7 Reduction of $\mu_p$-torsors and deformation data

Let $R$, $K$, and $k$ be as in §2.4, and let $\pi$ be a uniformizer of $R$. Assume further that $R$ contains the $p$th roots of unity. For any scheme or algebra $S$ over $R$, write $S_K$ and $S_k$

for its base changes to $K$ and $k$, respectively. Recall that we normalize the valuation of $p$ (not $\pi$) to be 1. Then $v(\pi) = 1/e$, where $e$ is the absolute ramification index of $R$. This is not always the standard convention, but it will be useful later in the thesis.

### 2.7.1 Reduction of $\mu_p$-torsors

We start with a result of Henrio:

**Proposition 2.32** ([Hen99], Chapter 5, Proposition 1.6)**.** *Let $X = \text{Spec } A$ be a flat affine scheme over $R$, with relative dimension $\leq 1$ and integral fibers. We suppose that $A$ is a factorial $R$-algebra which is complete with respect to the $\pi$-adic valuation. Let $Y_K \to X_K$ be a non-trivial, étale $\mu_p$-torsor, given by an equation $y^p = f$, where $f$ is invertible in $A_K$. Let $Y$ be the normalization of $X$ in $Y_K$; we suppose the special fiber of $Y$ is integral (in particular, reduced). Let $\eta$ (resp. $\eta'$) be the generic point of the special fiber of $X$ (resp. $Y$). The local rings $\mathcal{O}_{X,\eta}$ and $\mathcal{O}_{Y,\eta'}$ are thus discrete valuation rings with uniformizer $\pi$. Write $\delta$ for the valuation of the different of $\mathcal{O}_{Y,\eta'}/\mathcal{O}_{X,\eta}$. We then have two cases, depending on the value of $\delta$:*

- *If $\delta = 1$, then $Y = \text{Spec } B$, with $B = A[y]/(y^p - u)$, for $u$ a unit in $A$, unique up to multiplication by a pth power in $A^\times$. We say that the torsor $Y_K \to X_K$ has* multiplicative reduction.

- *If $0 \leq \delta < 1$, then $\delta = 1 - n(\frac{p-1}{e})$, where $n$ is an integer such that $0 < n \leq e/(p-1)$. Then $Y = \text{Spec } B$, with*

$$B = \frac{A[w]}{(\frac{(\pi^n w+1)^p - 1}{\pi^{pn}} - u)},$$

*for $u$ a unit of $A$. Also, $u$ is unique in the following sense: If an element $u' \in A^\times$ could take the place of $u$, then there exists $v \in A$ such that*

$$\pi^{pn}u' + 1 = (\pi^{pn}u + 1)(\pi^n v + 1)^p.$$

*If $\delta > 0$ (resp. $\delta = 0$), we say that the torsor $Y_K \to X_K$ has* additive reduction *(resp.* étale reduction*).*

**Remark 2.33.** (i) In [Hen99], Proposition 2.32 is stated for $X \to \text{Spec } R$ with dimension 1 fibers, but the proof carries over without change to the case of dimension 0 fibers as well (i.e., the case where $A$ is a discrete valuation ring containing $R$). It is this case that will be used in §2.7.2 to define deformation data.

(ii) The proof proceeds essentially by showing that, after multiplication by a $p$th power in $A_K$, we can choose $f$ to be in one of the following forms:

- $f \in A$ and the reduction $\overline{f}$ of $f$ in $A_k$ is not a $p$th power in $A_k$ (this is the case of multiplicative reduction).
- $f$ is of the form $1 + \pi^{pn}u$, where $n < e/(p-1)$, $u \in A$, and the reduction $\overline{u}$ of $u$ in $A_k$ is not a $p$th power in $A_k$ (this is the case of additive reduction).
- $f$ is of the form $1 + \pi^{pn}u$, where $n = e/(p-1)$, $u \in A$, and the reduction $\overline{u}$ of $u$ in $A_k$ is not of the form $x^p - x$ in $A_k$ (this is the case of étale reduction).

(iii) If $f$ is of the form $1+g$, where $v(g) > p/p-1$, then $f$ is a $p$th power in $A$. This can be seen, for instance, by using the binomial expansion of $\sqrt[p]{1+g}$, which converges because $A$ is $\pi$-adically complete.

(iv) In the cases of multiplicative and additive reduction, the extension $Y_k \to X_k$ is seen to be inseparable. In the case of étale reduction, it is an Artin-Schreier extension of the form $\overline{w}^p - \overline{w} = \overline{u}$, where $\overline{w}$ and $\overline{u}$ are the reductions of $w$ and $u$ to $B_k$ and $A_k$, respectively.

The following corollary will be used repeatedly in analyzing the stable reduction of covers (see §4.2, 4.3):

**Corollary 2.34.** *Assume that $R$ contains the $p^n$th roots of unity. Suppose $X = Spec\ A$, where $A = R\{T\}$. Let $Y_K \to X_K$ be a $\mu_{p^n}$-torsor given by the equation $y^{p^n} = f$, where $f = 1 + \sum_{i=1}^{\infty} a_i T^i$ such that $\min_i v(a_i) = n + \frac{1}{p-1}$ and $v(a_i) > n + \frac{1}{p-1}$ for all $i$ divisible by $p$. Define $h$ to be the largest $i$ such that $v(a_i) = n + \frac{1}{p-1}$. Then $Y_K \to X_K$ splits into a disjoint union of $p^{n-1}$ $\mu_p$-torsors, each with étale reduction birationally equivalent to an Artin-Schreier cover with conductor $h$.*

*Proof.* By Remark 2.33 (ii) and Lemma 2.14, we will be done if we can show that $f$ has a $p^{n-1}$st root $1 + au$ in $A$ such that $a \in R$, $v(a) = \frac{p}{p-1}$, and the reduction $\overline{u}$ of $u$ in $A_k = k[T]$ is of degree $h$ with only prime-to-$p$ degree terms. We can write $f = 1 + bw$ with $b \in R$ and $v(b) = n + \frac{1}{p-1}$. Suppose $n > 1$. Then, using the binomial theorem, a $p$th root of $f$ is given by

$$\sqrt[p]{f} = 1 + \frac{1/p}{1!}bw + \frac{(1/p)((1/p)-1)}{2!}(bw)^2 + \cdots .$$

Since $v(b) > \frac{p}{p-1}$, this series converges, and is in $A$. It can be written as $\sqrt[p]{f} = 1 + cx$, where $c = b/p \in R$, $v(c) = (n-1) + \frac{1}{p-1}$, and $x$ is congruent to $w$ mod $\pi$. We repeat this process $n-1$ times to obtain a $p^{n-1}$st root of $f$

$$\sqrt[p^{n-1}]{f} = 1 + au,$$

where $a \in R$ has $v(a) = 1 + \frac{1}{p-1} = \frac{p}{p-1}$ and $u \equiv w \pmod{\pi}$. If $n = 1$, then we skip the above process (i.e., we "repeat this process" zero times) and set $a = b$, $u = w$. By assumption, $w$ has degree $h$ and only prime-to-$p$ degree terms, and we are done. □

**Remark 2.35.** The above argument shows that if an element $r$ in $R$ can be written as $1 + w$, with $v(w) > n + \frac{1}{p-1}$, then $r$ has a $p^n$th root in $R$, which is congruent to $1 + \frac{w}{p^n}$ modulo $\pi$.

### 2.7.2 Deformation Data

Let $\overline{W}$ be any smooth proper curve over $k$. Let $H$ be a finite group and $\chi$ a 1-dimensional character $H \to \mathbb{F}_p^\times$. A *deformation datum* over $\overline{W}$ of type $(H, \chi)$ is an ordered pair $(\overline{V}, \omega)$ such that $\overline{V} \to \overline{W}$ is an $H$-Galois branched cover, $\omega$ is a meromorphic differential form on $\overline{V}$ that is either logarithmic or exact (i.e., $\omega = du/u$ or $du$ for $u \in k(\overline{V})$), and $\eta^*\omega = \chi(\eta)\omega$ for all $\eta \in H$. If $\omega$ is logarithmic (resp. exact), it is called multiplicative (resp. additive). When $\overline{V}$ is understood, we will sometimes speak of the deformation datum $\omega$.

If $(\overline{V}, \omega)$ is a deformation datum, and $w \in \overline{W}$ is a closed point, we define $m_w$ to be the order of the prime-to-$p$ part of the ramification index of $\overline{V} \to \overline{W}$ at $w$. Define $h_w$ to be $\mathrm{ord}_v(\omega) + 1$, where $v \in \overline{V}$ is any point which maps to $w \in \overline{W}$. This is well-defined because $\omega$ transforms nicely via $H$. Lastly, define $\sigma_x = h_w/m_w$. We call $w$ a *critical point* of the deformation datum $(\overline{V}, \omega)$ if $(h_w, m_w) \neq (1, 1)$. Note that every deformation datum contains only a finite number of critical points. The ordered pair $(h_w, m_w)$ is called the *signature* of $(\overline{V}, \omega)$ (or of $\omega$, if $\overline{V}$ is understood) at $w$, and $\sigma_w$ is called the *invariant* of the deformation datum at $w$.

**Proposition 2.36.** *If $N \subseteq \ker(\chi)$, then $(\overline{V}, \omega)$ descends naturally to a deformation datum $(\overline{V}/N, \omega')$. If, in addition, $p \nmid |N|$, then $(\overline{V}/N, \omega')$ has the same invariant at all points of $\overline{W}$ as $(\overline{V}, \omega)$.*

*Proof.* Since $\overline{V}$ is a smooth curve, $\Omega_{k(\overline{V})/k}$, the space of meromorphic differential forms on $\overline{V}$, is a one-dimensional $k(\overline{V})$-module. So we can choose any $t \in k(\overline{V})$, and write $\omega = f dt$ for some $f \in k(\overline{V})$. Let us choose $t \in k(\overline{V})^N$. Then, since $\omega$ is fixed by $N$, we must have $f \in k(\overline{V})^N$ as well. Since $k(\overline{V})^N = k(\overline{V}/N)$, $\omega$ can naturally be viewed as an element of $\Omega_{k(\overline{V}/N)/k}$, that is, a meromorphic differential on $\overline{V}/N$. It clearly satisfies the properties of a deformation datum. Thus we have descended $\omega$ to $\omega'$.

Now, assume $p \nmid |N|$. Let $v \in \overline{V}$ lie above a point $v' \in \overline{V}/N$ and above a point $w \in \overline{W}$. Let $(h_w, m_w)$ be the signature of $\omega$ at $w$, and let $(h'_w, m'_w)$ be the signature of $\omega'$ at $w$. Suppose the ramification index of $v$ in $\overline{V} \to \overline{V}/N$ is $\mu$. It is then clear that $m_w = \mu m'_w$. In a formal neighborhood of $v'$, we can use Kummer theory to write the cover $\overline{V} \to \overline{V}/N$ by the equation $k[[t]] \hookrightarrow k[[t]][\tau]/(\tau^\mu - t)$, where $t$ is a local parameter at $v'$ and $\tau$ is a local parameter at $v$. By the definition of $h'_w$, we can write $\omega' = (ct^{h'_w - 1} + \sum_{i=1}^{\infty} c_i t^{h'_w - 1 + i})dt$. In terms of $\tau$, $\omega$ can then be written as $(c\tau^{\mu(h'_w - 1)} + \sum_{i=1}^{\infty} c_i \tau^{\mu(h_w - 1) + i})\mu\tau^{\mu - 1} d\tau$. Then $h_w = \mu(h'_w - 1) + (\mu - 1) + 1 = \mu h'_w$. So $\sigma'_w = h'_w/m'_w = h_w/m_w = \sigma_w$. $\square$

**Proposition 2.37.** *Let $(\overline{V}, \omega)$ be a deformation datum of type $(H, \chi)$. Let $|H/\ker(\chi)| =$*

*$\mu$, and let $v \in \overline{V}$ be a tamely ramified point lying over $w \in \overline{W}$. Then $\sigma_w \in \frac{1}{\mu}\mathbb{Z}$.*

*Proof.* Let $I_v \subseteq H$ be the inertia group of $\phi : \overline{V} \to X$ at $v$. Then $|I_v/(I_v \cap \ker(\chi))| \mid \mu$. In a formal neighborhood of $v$, we can use Kummer theory to see that $\phi$ is given by the equation $k[[t]] \hookrightarrow k[[t]][\tau]/(\tau^{m_w} - t)$, where $t$ is a local parameter at $w$ and $\tau$ is a local parameter at $v$. Expanding $\omega$ out as a Laurent series in $\tau$, we can write

$$\omega = \left( c\tau^{h_w - 1} + \sum_{i=1}^{\infty} c_i \tau^{h_w - 1 + i} \right) d\tau.$$

Let $g$ be a generator of $I_v$ such that $g^*(\tau) = \zeta_{m_w}\tau$. Since $g^\mu \in \ker(\chi)$, we have that $(g^\mu)^*\omega = \omega$. Thus $(g^\mu)^*(\tau^{h_w-1}d\tau) = \tau^{h_w-1}d\tau$. So $\mu h_w$ is a multiple of $m_w$. Therefore, $\sigma_w = \frac{h_w}{m_w} \in \frac{1}{\mu}\mathbb{Z}$. $\square$

**Proposition 2.38** (Local vanishing cycles formula, cf. [Wew03b], p. 998)**.**

*(i) Suppose $(\overline{V}, \omega)$ is a deformation datum with $\overline{V} \to \overline{W}$ tamely ramified. Let $B$ be the set of critical points of $(\overline{V}, \omega)$. Suppose the genus of $\overline{W}$ is $g_W$. Then*

$$\sum_{b \in B} (\sigma_b - 1) = 2g_W - 2. \tag{2.7.1}$$

*(ii) Suppose we are in the situation of part (i), except that $\overline{V} \to \overline{W}$ is wildly ramified above one point $w \in \overline{W}$, such that the inertia group of a point $v$ above $w$ is $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m_w$ with $p \nmid m_w$. For $1 \leq i \leq n$, let $h_i$ be the ith lower jump of the extension $\hat{\mathcal{O}}_{\overline{V},v}/\hat{\mathcal{O}}_{\overline{W},w}$ (see §2.3), and let $\sigma_i = h_i/m_w$. We maintain the notation $(h_w, m_w)$ for the signature of $\omega$ at $w$, and $\sigma_w$ for the invariant at $w$ (note that there is not necessarily any relation between the $\sigma_i$ and $\sigma_w$). Let $B$ be the set of critical points*

*for $\omega$ other than $w$. Then we have*

$$\frac{\sigma_w}{p^n} - 1 - \sum_{i=1}^{n} \frac{p-1}{p^i}\sigma_i + \sum_{b \in B}(\sigma_b - 1) = 2g_W - 2. \tag{2.7.2}$$

*Proof. To (i):* Let $g_V$ be the genus of $\overline{V}$, and $d$ the degree of the map $\overline{V} \to \overline{W}$. By the Hurwitz formula,

$$2g_V - 2 = d(2g_W - 2) + d(\sum_{b \in B}(1 - \frac{1}{m_b})).$$

But $2g_V - 2$ is the degree of a differential form on $\overline{V}$, so $2g_V - 2 = \sum_{b \in B} \frac{d}{m_b}(h_b - 1)$. The local vanishing cycles formula then follows.

*To (ii):* Using Lemma 2.11 (i), the Hurwitz formula this time yields

$$2g_V - 2 = d(2g_W - 2) + d\sum_{b \in B}(1 - \frac{1}{m_b}) + \frac{d}{p^n m_w}(p^n m_w - 1 + \sum_{i=1}^{n} h_i p^{n-i}(p-1)).$$

Furthermore, the degree of a differential form on $\overline{V}$ is

$$\left(\sum_{b \in B} \frac{d}{m_b}(h_b - 1)\right) + \frac{d}{p^n m_w}(h_w - 1).$$

Substituting this for $2g_V - 2$ and rearranging yields the formula. □

Deformation data arise naturally from the stable reduction of covers. Say $f : Y \to X$ is a branched $G$-cover as in §2.5, with stable model $f^{st} : Y^{st} \to X^{st}$ defined over $K^{st}$ and stable reduction $f : \overline{Y} \to \overline{X}$. Much information is lost when we pass from the stable model to the stable reduction, and deformation data provide a way to retain some of this information. This process is described in detail in [Hen99, 5, §1] in the case where the inertia group of a component has order $p$. We generalize it here to the case where the inertia group is cyclic of order $p^r$. For this construction, we can replace $K^{st}$ with as large a finite extension as we wish. In particular, we assume that $K^{st}$ contains a $p$th root of unity.

**Construction 2.39.** Let $\overline{V}$ be an irreducible component of $\overline{Y}$ with generic point $\eta$ and nontrivial generic inertia group $I \cong \mathbb{Z}/p^r \subset G$. Write $B = \hat{\mathcal{O}}_{Y^{st},\eta}$, and $C = B^I$, the invariants of $B$ under the action of $I$. Then $B$ (resp. $C$) is a complete, mixed characteristic, discrete valuation ring with residue field $k(\overline{V})$ (resp. $k(\overline{V})^{p^r}$). $I \cong \mathbb{Z}/p^r$ acts on $B$; for $0 \leq i \leq r$, we write $I_i$ for the subgroup of order $p^i$ in $I$, and we write $C_i$ for the fixed ring $B^{I_{r-i+1}}$. Thus $C_1 = C$. Then for $1 \leq i \leq r$, the extension $C_i \hookrightarrow C_{i+1}$ is an extension of complete discrete valuation rings satisfying the conditions of Proposition 2.32 but with relative dimension 0 instead of 1 over $R^{st}$ (see Remark 2.33 (i)). On the generic fiber, it is given by an equation $y^p = z$, where $z$ is well-defined up to raising to a prime-to-$p$ power in $C_i^{\times}/(C_i^{\times})^p$. We make $z$ completely well-defined in $C_i^{\times}/(C_i^{\times})^p$ by fixing a $p$th root of unity $\mu$ and a generator $\alpha$ of $\text{Aut}(C_{i+1}/C_i)$ and forcing $\alpha(z) = \mu z$. In both the case of multiplicative and additive reduction, Proposition 2.32 yields an element

$$\overline{u} \in C_i \otimes_{R^{st}} k = k(\overline{V})^{p^{r-i+1}} \cong k(\overline{V})^{p^r},$$

the last isomorphism coming from raising to the $p^{i-1}$st power. In the case of multiplicative reduction, write $\omega_i = d\overline{u}/\overline{u}$, and in the case of additive reduction, write $\omega_i = d\overline{u}$. In both cases, $\omega_i$ can be viewed as a differential form on $k(\overline{V})^{p^r}$. Write $\overline{V}'$ for the curve whose function field is $C \otimes_{R^{st}} k = k(\overline{V})^{p^r} \subset k(\overline{V})$. Then each $\omega_i$ is a meromorphic differential form on $\overline{V}'$.

Furthermore, let $D$ be the decomposition group of $\overline{V}$, and write $H = D/I$. Then if $\overline{W}$ is the component of $\overline{X}$ lying below $\overline{V}$, we have maps $\overline{V} \to \overline{V}' \to \overline{W}$, with $\overline{W} = \overline{V}'/H$. Any $\eta \in H$ has a canonical conjugation action on $I$, and also on the subquotient $J_i$ of $I$ given by $I_{r-i+1}/I_{r-i}$. This action is given by a homomorphism $\chi : H \to (\mathbb{F}_p)^\times$. We claim to have constructed, for each $i$, a deformation datum $(\overline{V}', \omega_i)$ of type $(H, \chi)$ over $\overline{W}$.

Everything is clear except for the transformation property, so let $\eta \in H$. Then for $z$ as in the construction, taking a $p$th root of $z$ and of $\eta^*z$ must yield the same extension, so $\eta^*z = c^p z^q$ with $c \in C_i$ and $q \in \{1, \ldots, p-1\}$. It follows that $\eta^*y = \zeta c y^q$ for $\zeta$ some $p$th root of unity. It also follows that $\eta^*(\omega_i) = q\omega_i$. Let $\alpha$ be a generator of $J_i$. We must show that $\eta\alpha\eta^{-1} = \alpha^q$.

Write $\alpha^*y = \mu y$ for some, possibly different, $p$th root of unity $\mu$. Then

$$(\eta\alpha\eta^{-1})^*(y) = (\eta^{-1})^*\alpha^*\eta^*y = (\eta^{-1})^*\alpha^*\zeta c y^q = (\eta^{-1})^*\mu^q\zeta c y^q = \mu^q y.$$

Thus $\eta\alpha\eta^{-1} = \alpha^q$, and we are done. This completes Construction 2.39.

For the rest of this section, we will only concern ourselves with deformation data that arise from the stable reduction of branched $G$-covers $Y \to X = \mathbb{P}^1$ where $G$ has a cyclic $p$-Sylow subgroup, via Construction 2.39. We will use the notations of §2.5 and Construction 2.39 throughout this section. Note that in the case of such a cover, we have introduced a collection of deformation data for each irreducible component of $\overline{Y}$ with nontrivial inertia. The size of the collection is $r$, where the size of the inertia group is $p^r$. We will sometimes call the deformation datum $(\overline{V}', \omega_1)$ (resp. the differential form $\omega_1$)

the *bottom deformation datum* (resp. the *bottom differential form*) for $\overline{V}'$.

From [Wew03b, Proposition 1.7], we have the following result in the case of inertia groups of order $p$. The proof is the same in our case:

**Proposition 2.40.** *Say $(\overline{V}', \omega)$ is a deformation datum arising from the stable reduction of a cover as in Construction 2.39, and let $\overline{W}$ be the component of $\overline{X}$ lying under $\overline{V}'$. Then a critical point $x$ of the deformation datum on $\overline{W}$ is either a singular point of $\overline{X}$ or the specialization of a branch point of $Y \to X$ with ramification index divisible by $p$. In the first case, $\sigma_x \neq 0$, and in the second case, $\sigma_x = 0$ and $\omega$ is logarithmic.*

We should also note the easy fact that any deformation datum $(\overline{V}, \omega)$ such that $\omega$ has a simple pole (equivalently, there is a critical point with $\sigma = 0$) is multiplicative, whereas any deformation datum $(Z, \omega)$ such that $\omega$ has a multiple pole (equivalently, there is a critical point where $\sigma$ is negative) is additive. This is a property of differential forms in characteristic $p$, and has nothing to do with the transformation property.

**Proposition 2.41.** *Let $(\overline{V}', \omega_1)$ be the bottom deformation datum for some irreducible component $\overline{V}$ of $\overline{Y}$. If $\omega_1$ is logarithmic, then so are all $\omega_i$'s arising from Construction 2.39. Furthermore, $\omega_i = \omega_1$ for all $i$.*

*Proof.* As was mentioned before Construction 2.39, we may assume that $K^{st}$ contains the $p^r$th roots of unity. By Kummer theory, we can write $B \otimes_{R^{st}} K^{st} = (C \otimes_{R^{st}} K^{st})[\theta]/(\theta^{p^r} - \theta_1)$. After a further extension of $K^{st}$, we can assume $v(\theta_1) = 0$.

If $\omega_1$ is logarithmic, then the reduction $\overline{\theta}_1$ of $\theta_1$ to $k$ is not a $p$th power in $C \otimes_{R^{st}} k$. We thus know that $\omega_1 = d\overline{\theta}_1/\overline{\theta}_1$. It is then easy to see that $\omega_i$ arises from the equation

$y^p = \theta_i$ where $\theta_i = \sqrt[p^{i-1}]{\theta_1}$. Under the $p^{i-1}$st power isomorphism $\iota : C_i \otimes_{R^{st}} k \to C \otimes_{R^{st}} k$, $\iota(\theta_i) = \theta_1$. So $\omega_i$ is logarithmic, and is equal to $\frac{d\theta_1}{\theta_1}$, which is equal to $\omega_1$. □

The next two results, Propositions 2.42 and 2.44, each generalize one part of the theorem [Hen99, 5, Theorem 1.10]. They relate deformation data, conductors of Artin-Schreier covers, épaisseurs of annuli, and differents of extensions. They are extremely important in what follows, as they are the "guts" behind the cleaner interface provided by Lemmas 2.46 and 2.47. The proof of Theorem 1.4 depends heavily on Lemmas 2.46 and 2.47. First we must set up some notation.

Suppose $\overline{W}$ and $\overline{W}'$ are intersecting components of $\overline{X}$, and let $\overline{V}$ and $\overline{V}'$ be intersecting components of $\overline{Y}$ lying above them. Let $x$ be the intersection point of $\overline{W}$ and $\overline{W}'$, and let $y$ be a point of $\overline{V} \cap \overline{V}'$ above $x$. By Corollary 2.18, assume without loss of generality that the inertia group of $\overline{V}$ contains that of $\overline{V}'$. Let $I \cong \mathbb{Z}/p^r$ (resp. $I' \cong \mathbb{Z}/p^{r'}$) be the inertia group of $\overline{V}$ (resp. $\overline{V}'$). By Proposition 2.22, $r \geq 1$. For each $i$, $1 \leq i \leq r$, there is a deformation datum with differential form $\omega_i$ associated to $\overline{V}$. For each $i'$, $1 \leq i' \leq r'$, there is a deformation datum $\omega'_{i'}$ associated to $\overline{V}'$. Let $m_x$ be the prime-to-$p$ part of the ramification index at $x$. The inclusion $\hat{\mathcal{O}}_{\overline{W}',x} \hookrightarrow \hat{\mathcal{O}}_{\overline{V}',y}$ induced from the cover is a composition

$$\hat{\mathcal{O}}_{\overline{W}',x} \hookrightarrow S \hookrightarrow \hat{\mathcal{O}}_{\overline{V}',y}$$

where $\hat{\mathcal{O}}_{\overline{W}',x} \hookrightarrow S$ is a totally ramified Galois extension with group $I \cong \mathbb{Z}/p^{r-r'} \rtimes \mathbb{Z}/m_x$ and $S \hookrightarrow \hat{\mathcal{O}}_{\overline{V}',y}$ is a purely inseparable extension of degree $p^{r'}$. Let $J$ be the inertia group of $y$ in $G$, and let $J_i$ (resp. $I_i$) be the unique subgroup of order $p^i$ in $J$ (resp. $I$). The following proposition gives a compatibility between deformation data, and also relates deformation data to the geometry of $\overline{Y}$.

**Proposition 2.42.** *With $x$ as above, let $(h_{i,x}, m_x)$ (resp. $(h'_{i',x}, m_x)$) be the signature of $\omega_i$ (resp. $\omega'_{i'}$) at $x$. Write $\sigma_{i,x} = h_{i,x}/m_x$ and $\sigma'_{i',x} = h'_{i',x}/m_x$. Then the following hold:*

*(i) If $i = i' + r - r'$, then $h_{i,x} = -h'_{i',x}$ and $\sigma_{i,x} = -\sigma'_{i',x}$.*

*(ii) If $i \leq r - r'$, then $h_{i,x} = h$, where $h$ is the upper (equivalently lower) jump in the extension $S^{I_{r-r'-i+1}} \hookrightarrow S^{I_{r-r'-i}}$. Also, $\sigma_{i,x} = \sigma$, where $\sigma$ is the upper jump in the extension $S^{I_{r-r'-i+1} \rtimes \mathbb{Z}/m_x} \hookrightarrow S^{I_{r-r'-i}}$.*

*Proof.* (cf. [Wew03b, Proposition 1.8]) The group $J$ acts on the annulus $\mathcal{A} = \text{Spec } \hat{\mathcal{O}}_{Y^{st},y}$. The proposition follows from [Hen99, 5, Proposition 1.10] applied to the formal annulus $\mathcal{A}/(J_{r-i+1})$ and an automorphism given by a generator of $J_{r-i+1}/J_{r-i}$ considered as a subquotient of $J$. The statement about $\sigma_{i,x}$ follows by dividing both sides of the equation $h_{i,x} = h$ by $m_x$. Note that what we call $h_{i,x}$, Henrio calls $-m$. □

**Remark 2.43.** Consider the $\mathbb{Z}/p^{r-r'} \rtimes \mathbb{Z}/m_x$-extension $\hat{\mathcal{O}}_{\overline{W}',x} \hookrightarrow S$. If the $j_i$ are its lower jumps (see §2.3), then Proposition 2.42, combined with [OP08, Lemma 3.1], shows that $j_i = h_{i,x}$. By Remark 2.12, the conductor of this extension is equal to

$$\left(\sum_{i=1}^{r-r'-1} \frac{p-1}{p^i}\sigma_{i,x}\right) + \frac{1}{p^{r-r'-1}}\sigma_{r-r',x}.$$

In this same context, let $\delta_i$ be the valuation of the different of the extension $C_i \hookrightarrow C_{i+1}$ giving rise to $\omega_i$ (we will call $\delta_i$ the different *corresponding to* $\omega_i$). Likewise, let $\delta'_{i'}$ be the valuation of the different of the extension $C'_{i'} \hookrightarrow C'_{i'+1}$ giving rise to $\omega'_{i'}$. We have the following proposition relating the change in the differents and the épaisseur of the annulus corresponding to $x$:

**Proposition 2.44.** *Assume the notations of Proposition 2.42. Let $\epsilon_x$ be the épaisseur of the formal annulus corresponding to $x$.*

- *If $i = i' + r - r'$, then $\delta_i - \delta'_{i'} = \frac{\epsilon_x \sigma_{i,x}(p-1)}{p^i}$.*

- *If $i \leq r - r'$, then $\delta_i = \frac{\epsilon_x \sigma_{i,x}(p-1)}{p^i}$.*

*Proof.* As in the proof of Proposition 2.42, let $\mathcal{A} = \text{Spec } \hat{\mathcal{O}}_{Y^{st},y}$. If $\epsilon$ is the épaisseur of $\mathcal{A}/(J_{r-i+1})$, then [Hen99, 5, Proposition 1.10] shows that $\delta_i - \delta'_{i'} = \epsilon h_{i,x}(p-1)$ in the case $i = i' + r - r'$ and $\delta_i - 0 = \epsilon h_{i,x}(p-1)$ in the case $i < r - r'$. [Ray99, Proposition 2.3.2 (a)] shows that $\epsilon_x = p^i m_x \epsilon$. The proposition follows. □

The above propositions require a great deal of notation. But the quantities which we define now will encapsulate most of the information we need.

**Definition 2.45.** Let $\overline{W}$ be a $p^r$-component of $\overline{X}$, and let $\omega_i$, $1 \leq i \leq r$, be the deformation data above $\overline{W}$.

- For any $w \in \overline{W}$, define the *effective invariant* $\sigma_w^{\text{eff}}$ by

$$\sigma_w^{\text{eff}} = \left( \sum_{i=1}^{r-1} \frac{p-1}{p^i} \sigma_{i,w} \right) + \frac{1}{p^{r-1}} \sigma_{r,w}.$$

  Note that this is a weighted average of the $\sigma_i$'s.

- Define the *effective different* $\delta^{\text{eff}}$ by

$$\delta^{\text{eff}} = \left(\sum_{i=1}^{r-1} \delta_i\right) + \frac{p}{p-1}\delta_r.$$

**Lemma 2.46.** *Assume the notations of Proposition 2.44. Let $\sigma_x^{\mathit{eff}}$ be the effective invariant at $x$ of the deformation data above $\overline{W}$. Let $\delta^{\mathit{eff}}$ (resp. $(\delta')^{\mathit{eff}}$) be the effective different above $\overline{W}$ (resp. $\overline{W}'$). Then*

$$\delta^{\mathit{eff}} - (\delta')^{\mathit{eff}} = \sigma^{\mathit{eff}}\epsilon_x.$$

*Proof.* We sum the equations from Proposition 2.44 for $1 \leq i \leq r-1$. Then we add $\frac{p}{p-1}$ times the equation for $i = r$. This exactly gives $\delta^{\text{eff}} - (\delta')^{\text{eff}} = \sigma^{\text{eff}}\epsilon_x$. □

The following lemma will be used repeatedly in Chapter 4:

**Lemma 2.47.** *Let $\overline{W}$ be an inseparable component of $\overline{X}$ which is not a tail, and let $w \in \overline{W}$ be a singular point of $\overline{X}$ such that $\overline{W} \prec w$. Suppose that $\overline{f}$ is monotonic from $\overline{W}$. Denote by $\sigma_w^{\mathit{eff}}$ the effective invariant for the deformation data above $\overline{W}$ at $w$. Let $\Pi$ be the set of branch points of $f$ with branching index divisible by $p$ that specialize outward from $w$. Let $B$ index the set of étale tails $\overline{X}_b$ lying outward from $w$. Then the following formula holds:*

$$\sigma_w^{\mathit{eff}} - 1 = \sum_{b \in B} (\sigma_b - 1) - |\Pi|.$$

*Proof.* In the context of this proof, call a set $S$ of singular points of $\overline{X}$ *admissible* for $w$ if the following hold:

- For each $s \in S$, $w \preceq s$.

- For each tail $\overline{X}_b \succ \overline{W}$, there exists exactly one $s \in S$ such that $s \prec \overline{X}_b$.

Clearly, the set $\{w\}$ is admissible for $w$, as is the set $S_{\max}$ containing all of the intersection points of tails of $\overline{X}$ lying outward from $w$ with the interior of $\overline{X}$. Suppose $w' \succeq w$ is a singular point of $\overline{X}$ lying on the intersection of two components $\overline{W}' \prec \overline{W}''$. Then we write $\sigma_{i,w'}$ (resp. $\sigma_{w'}^{\text{eff}}$) to mean the invariant for the $i$th differential form (resp. the effective invariant) at $w'$ for the deformation data above $\overline{W}'$. For an admissible set $S$ for $w$, write $\Pi_S$ for the set of branch points of $f$ with branching index divisible by $p$ that specialize outward from some element of $S$. We will prove the lemma by proving the stronger statement that for all admissible $S$ for $w$,

$$\sum_{s \in S} (\sigma_s^{\text{eff}} - 1) + |\Pi_S| = \sum_{b \in B} (\sigma_b - 1). \tag{2.7.3}$$

The lemma is exactly (2.7.3) for $S = \{w\}$.

We will first prove (Step 1) that $F(S) := \sum_{s \in S} (\sigma_s^{\text{eff}} - 1) + |\Pi_S|$ is independent of $S$. We will then prove (Step 2) that for $S = S_{\max}$, (2.7.3) holds, thus proving the lemma.

*Step 1:* We show that $F(S)$ is independent of $S$ by "outward induction on $S$." Suppose we are given an admissible set $S$ for $w$. Pick some $s_0 \in S$. Then $s_0$ lies on the intersection of two components $\overline{L} \prec \overline{M}$. Let $T$ be the set of singular points of $\overline{X}$ lying on $\overline{M}$, excluding $s_0$. Then $S \cup T \backslash \{s_0\}$ is also admissible for $w$. It is clear that every admissible $S$ can be obtained from $\{w\}$ by a repetition of this process. Let $\alpha_{\overline{M}} = 1$ if there is a branch point of $f$ with branching index divisible by $p$ specializing to $\overline{M}$. Otherwise, $\alpha_{\overline{M}} := 0$. We need to show that $F(S) = F(S \cup T \backslash \{s_0\})$; in other words, that $\sigma_{s_0}^{\text{eff}} - 1 = \sum_{t \in T} (\sigma_t^{\text{eff}} - 1) - \alpha_{\overline{M}}$.

We will use the local vanishing cycles formulas (2.7.1) and (2.7.2). First, assume that $\overline{L}$ and $\overline{M}$ are both $p^r$-components, for some common $r$. Then, for each deformation datum $\omega_i$ above $\overline{M}$, $1 \leq i \leq r$, the local vanishing cycles formula (2.7.1) combined with Propositions 2.40 and 2.42 yields

$$(-\sigma_{i,s_0} - 1) + \sum_{t \in T} (\sigma_{i,t} - 1) - \alpha_{\overline{M}} = -2. \tag{2.7.4}$$

In other words,

$$\sigma_{i,s_0} - 1 = \sum_{t \in T} (\sigma_{i,t} - 1) - \alpha_{\overline{M}}. \tag{2.7.5}$$

For $1 \leq i \leq r-1$, we multiply the $i$th equation (2.7.5) by $\frac{p-1}{p^i}$ to obtain an equation $E_i$. For $i = r$, we multiply it by $\frac{1}{p^{r-1}}$ to obtain $E_r$. Note that these coefficients add up to 1. Then we add up the equations $E_i$. By the definition of the effective invariant and the fact that these coefficients add up to 1, we obtain $\sigma_{s_0}^{\text{eff}} - 1 = \sum_{t \in T} (\sigma_t^{\text{eff}} - 1) - \alpha_{\overline{M}}$.

Now assume that $\overline{L}$ is a $p^r$-component and $\overline{M}$ is a $p^{r-j}$-component. Then, for each deformation datum $\omega_i$ above $\overline{M}$, $1 \leq i \leq r-j$, the local vanishing cycles formula (2.7.2) combined with Propositions 2.40 and 2.42 yields

$$\left( -\frac{\sigma_{i+j,s_0}}{p^j} - 1 - \sum_{\alpha=1}^{j} \frac{p-1}{p^\alpha} \sigma_{\alpha,s_0} \right) + \sum_{t \in T} (\sigma_{i,t} - 1) - \alpha_{\overline{M}} = -2. \tag{2.7.6}$$

In other words,

$$\frac{\sigma_{i+j,s_0}}{p^j} - 1 + \sum_{\alpha=1}^{j} \frac{p-1}{p^\alpha} \sigma_{\alpha,s_0} = \sum_{t \in T} (\sigma_{i,t} - 1) - \alpha_{\overline{M}} \tag{2.7.7}$$

For $1 \leq i \leq r-j-1$, we multiply the $i$th equation (2.7.7) by $\frac{p-1}{p^i}$ to obtain an equation $E_i$. For $i = r-j$, we multiply it by $\frac{1}{p^{r-j-1}}$ to obtain $E_{r-j}$. Note that these coefficients add up to 1. Then we add up the equations $E_i$. Again, by the definition of the effective invariant and the fact that these coefficients add up to 1, a straightforward calculation shows that we obtain $\sigma_{s_0}^{\text{eff}} - 1 = \sum_{t \in T}(\sigma_t^{\text{eff}} - 1) - \alpha_{\overline{M}}$. We have shown that $F(S)$ is independent of $S$.

*Step 2:* We calculate $F(S_{\max})$. Write $S_{\max} = S_{\text{ét}} \cup S_{\text{insep}}$, where $S_{\text{ét}}$ is the set of points of $S$ lying on the intersection of an étale tail with the rest of $\overline{X}$, and $S_{\text{insep}}$ is the set of points in $S$ lying on the intersection of an inseparable tail with the rest of $\overline{X}$. If $s_b \in S_{\text{ét}}$ lies on the intersection of a $p^r$-component and an étale tail $\overline{X}_b$, then $\sigma_{s_b}^{\text{eff}} = \left(\sum_{i=1}^{r-1} \frac{p-1}{p^i}\sigma_{i,s_b}\right) + \frac{1}{p^{r-1}}\sigma_{r,s_b}$. Proposition 2.42, combined with Remark 2.43, shows that this is equal to $\sigma_b$.

Now suppose $s \in S_{\text{insep}}$ lies on the intersection of a $p^r$-component and an inseparable tail $\overline{X}_s$ which is a $p^{r-j}$-component. Then for each deformation datum $\omega_i$ above $\overline{X}_s$, $1 \leq i \leq r-1$, (2.7.2) shows that

$$\left(-\frac{\sigma_{i+j,s}}{p^j} - 1 - \sum_{\alpha=1}^{j} \frac{p-1}{p^\alpha}\sigma_{\alpha,s}\right) - \alpha_{\overline{X}_s} = -2. \tag{2.7.8}$$

In other words:

$$\frac{\sigma_{i+j,s}}{p^j} + \sum_{\alpha=1}^{j} \frac{p-1}{p^\alpha}\sigma_{\alpha,s} = 1 - \alpha_{\overline{X}_s}. \tag{2.7.9}$$

Again, for $1 \leq i \leq r-j-1$, we multiply the $i$th equation (2.7.9) by $\frac{p-1}{p^i}$ to obtain an equation $E_i$. For $i = r-j$, we multiply it by $\frac{1}{p^{r-j-1}}$ to obtain $E_{r-j}$. As before, we add up the equations $E_i$ to obtain $\sigma_s^{\text{eff}} = 1 - \alpha_{\overline{X}_s}$.

Thus,

$$F(S_{\max}) = \sum_{b \in B}(\sigma_b - 1) - \sum_{s \in S_{\max}}(\alpha_{\overline{X}_s}) + |\Pi_{S_{\max}}| = \sum_{b \in B}(\sigma_b - 1).$$

This is Equation (2.7.3), so we are done. □

# Chapter 3

# Combinatorial Properties of the Stable Reduction

Assume the notations of §2.5. In this chapter, $G$ is a finite group with a cyclic $p$-Sylow subgroup $P$ of order $p^n$. Recall that $m_G = |N_G(P)/Z_G(P)|$, and that we will write $m$ instead of $m_G$ when $G$ is understood. Also, $f^{aux} : Y^{aux} \to X$ (resp. $f^{str} : Y^{str} \to X$) is the auxiliary cover (resp. the strong auxiliary cover) with Galois group $G^{aux}$ (resp. $G^{str}$); see §2.6. Let $Q$ be the normal subgroup of order $p$ in $G^{str}$ (Proposition 2.29).

## 3.1 Vanishing cycles formulas

The vanishing cycles formula ([Ray99, Théorème 3.3.5]) is a key formula that helps us understand the structure of the stable reduction of a branched $G$-cover of curves in the case where $p$ exactly divides the order of $G$. Here, we generalize the formula to the case where $G$ has a cyclic $p$-Sylow group of arbitrary order. For any étale tail $\overline{X}_b$, recall that $m_b$ is the index of tame inertia at the point of intersection $x_b$ of $\overline{X}_b$ with the rest of $\overline{X}$, and $\sigma_b$ is the generalized ramification invariant (Definition 2.25). Set $h_b = m_b\sigma_b$. By Lemma 2.26, $h_b$ is an integer.

**Theorem 3.1.** *Let $f : Y \to X$, $X$ not necessarily $\mathbb{P}^1$, be a $G$-Galois cover as in §2.5, where $G$ has a cyclic $p$-Sylow subgroup. Let $B_0$ be the set of branch points. As in §2.5, there is a smooth model $X_R$ of $X$ where the specializations of the branch points do not collide, $f$ has bad reduction, and $\overline{f} : \overline{Y} \to \overline{X}$ is the stable reduction of $f$. Let $\Pi \subset B_0$ be the set of branch points which have branching index divisible by $p$. Let $B_{new}$ be an indexing set for the new tails and let $B_{prim}$ be an indexing set for the primitive tails. Let $B_{ét} = B_{new} \cup B_{prim}$. Let $g_X$ be the genus of $X$. Then we have the formula*

$$2g_X - 2 + |\Pi| = \sum_{b \in B_{new}} (\sigma_b - 1) + \sum_{b \in B_{prim}} (\sigma_b - 1). \tag{3.1.1}$$

Theorem 3.1 has the immediate corollary:

**Corollary 3.2.** *Assume further that $f$ is a three-point cover of $\mathbb{P}^1$. Then*

$$1 = \sum_{b \in B_{new}} (\sigma_b - 1) + \sum_{b \in B_{prim}} \sigma_b. \tag{3.1.2}$$

*Proof* (of the theorem). Write $|G^{str}| = p^{\nu}m$, where $p \nmid m$. Write $\tilde{Y}^{str} = Y^{str}/Q$. Let $g$ (resp. $\tilde{g}$) be the genus of $Y^{str}$ (resp. $\tilde{Y}^{str}$). Also, let $\Pi_i$ be the set of points of $X$ which are branched of index $p^i$ in $Y^{str} \to X$. Thus, $\Pi = \bigcup_{i=1}^{\nu} \Pi_i$. The Hurwitz formula applied to $Y^{str} \to X$ and $\tilde{Y}^{str} \to X$ respectively gives us the following two equalities:

$$2g - 2 = p^{\nu} m \left( 2g_X - 2 + \sum_{b \in B_{\text{ét}}} (1 - \frac{1}{m_b}) + \sum_{i=1}^{\nu} |\Pi_i|(1 - \frac{1}{p^i}) \right) \tag{3.1.3}$$

and

$$2\tilde{g} - 2 = p^{\nu-1} m \left( 2g_X - 2 + \sum_{b \in B_{\text{ét}}} (1 - \frac{1}{m_b}) + + \sum_{i=1}^{\nu} |\Pi_i|(1 - \frac{1}{p^{i-1}}) \right). \tag{3.1.4}$$

Since all of our schemes are flat over $R$, $p_a(\overline{Y}^{str}) = g$ and $p_a(\overline{\tilde{Y}}^{str}) = \tilde{g}$. Now, note that when we quotient out by $Q$, the morphism is radicial on the special fiber except above the tails, as $Q$ is contained in the inertia group of every irreducible component lying above the interior of $\overline{X}$. This means that the contribution from components above the interior of $\overline{X}$ to $p_a(\overline{Y}^{str})$ is the same as their contribution to $p_a(\overline{\tilde{Y}}^{str})$ Above each tail $\overline{X}_b$, $\overline{Y}^{str}$ consists of irreducible components $\overline{Y}_b$, each of which is the composition of a cyclic cover of order $m_b$ branched at two points $\infty_b$ and $0_b$ with a cyclic $p$-power cover totally ramified at the unique point of $\overline{Y}_b$ lying above $\infty_b$. Then Corollary 2.16 with group $G^{str}$ of order $p^{\nu} m$ applies to each tail $\overline{X}_b$. We obtain that

$$g - \tilde{g} = \frac{1}{2} p^{\nu-1}(p-1) m \left( \sum_{b \in B_{\text{ét}}} \left( (1 - \frac{1}{m_b}) + (\sigma_b + 1) - 2 \right) \right). \tag{3.1.5}$$

But subtracting the difference of (3.1.3) and (3.1.4) and dividing by two, we have

$$g - \tilde{g} = \frac{1}{2} p^{\nu-1}(p-1) m \left( 2g_X - 2 + \sum_{b \in B_{\text{ét}}} (1 - \frac{1}{m_b}) + \sum_{i=1}^{\nu} |\Pi_i| \right). \tag{3.1.6}$$

Setting the right-hand sides of (3.1.5) and (3.1.6) equal yields

$$2g_X - 2 + |\Pi| = \sum_{b \in B_{\text{ét}}} (\sigma_b - 1).$$

□

**Remark 3.3.** Since the upper numbering is invariant under quotients, it is clear that quotienting out by a normal subgroup of prime-to-$p$ order does not affect the quantity $\sigma_b$. Thus the quantity $\sigma_b$ as we have calculated it is not different than if we had calculated it using the standard auxiliary cover, rather than the strong auxiliary cover.

The above formula can be generalized. Recall that for every $i$, $1 \leq i \leq n$, we write $\Pi_i$ for the set of branch points of $f$ which have branching index divisible by $p^i$. Let $B_{d,e}$ be an indexing set for the $p^d$-components of $\overline{X}$ which intersect a $p^e$-component, $e > d$, and such that $\overline{f}$ is monotonic from the $p^d$-component. For $b \in B_{d,e}$, let $x_b$ be the (unique) point of intersection of the $p^d$-component indexed by $b$ with a $p^e$-component. Let $y_b$ be a point lying above $x_b$, and let $\mathcal{O}_{z_b} = \mathcal{O}_{\overline{Y},y_b}^{\mathbb{Z}/p^d}$. Finally, let $\sigma_{i,b}$ be the $i$th *upper* jump for the extension $\mathcal{O}_{\overline{X},x_b} \hookrightarrow \mathcal{O}_{z_b}$. If $i = e - d$, we will sometimes just write $\sigma_b$ for $\sigma_{i,b}$. Then we have the following formula:

**Proposition 3.4.** *For each $r$, $0 \leq r \leq n-1$, such that there exists some nonempty $B_{d,e}$ with $d \leq r < e$,*

$$2g_X - 2 + |\Pi_{r+1}| \geq \sum_{\substack{d,e \\ d \leq r < e}} \sum_{b \in B_{d,e}} (\sigma_{e-r,b} - 1). \tag{3.1.7}$$

*If $f$ is monotonic, we have equality in (3.1.7).*

*Proof.* Again, consider the strong auxiliary cover $f^{str}$, which has Galois group $G^{str} \cong \mathbb{Z}/p^{\nu} \rtimes \mathbb{Z}/m$ for some $\nu > r$ and stable reduction $\overline{f}^{str}$. Let $(\overline{f}^{str})'$ be the quotient of the cover $\overline{f}^{str}$ by the unique subgroup $Q_r \subset G^{str}$ isomorphic to $\mathbb{Z}/p^r$. Also, let $Q_{r+1}$ be the unique subgroup of $G^{str}$ isomorphic to $\mathbb{Z}/p^{r+1}$. Let $g$ be the genus of $Y^{str}/Q_r$ and let $\tilde{g}$

be the genus of $Y^{str}/Q_{r+1}$. Then, as in the proof of Theorem 3.1,

$$g - \tilde{g} = \frac{1}{2}p^{\nu-r-1}(p-1)m\left(2g_X - 2 + \sum_{b \in B_{\text{ét}}}\left(1 - \frac{1}{m_b}\right) + |\Pi_{r+1}|\right). \tag{3.1.8}$$

Since $\overline{f}$ is monotonic from each component indexed by $B_{d,e}$, each component lying outward from $B_{d,e}$ and each component in $B_{d,e}$ is étale in the special fiber of $\overline{Y}^{str}/Q_r$. Also, any prime-to-$p$ branch point of $f^{str}/Q_r$ specializes either to or outward from a component indexed by $B_{d,e}$. Above the unique wildly branched point of $\overline{X}_b$ for any $b \in B_{d,e}$, the conductor of higher ramification for $\overline{f'}$ is $\sigma_{e-r,b}$. We note that quotienting out the cover $\overline{f'}$ by the subgroup $Q_{r+1}/Q_r < G^{str}/Q_r$ of order $p$ cannot increase the contribution to the arithmetic genus from those components of $\overline{Y}^{str}/Q_r$ lying above components of $\overline{X}$ which do not lie outward from some component indexed by $B_{d,e}$. In fact, if $\overline{f}$ is monotonic, then the action of $Q_{r+1}/Q_r$ is radicial on these components and does not affect their contribution to the arithmetic genus at all.

By flatness, we have $g = p_a(\overline{Y}^{str}/Q_r)$ and $\tilde{g} = p_a(\overline{Y}^{str}/Q_{r+1})$. Now, we apply Corollary 2.16 to obtain

$$g - \tilde{g} \geq \frac{1}{2}p^{\nu-r-1}(p-1)m\left(\sum_{b \in B_{d,e}}(\sigma_{e-r,b} + 1 - 2) + \sum_{b \in B_{\text{ét}}}\left(1 - \frac{1}{m_b}\right)\right), \tag{3.1.9}$$

with equality if $\overline{f}$ is monotonic. Combining (3.1.8) and (3.1.9) yields the proposition. □

**Remark 3.5.** We have some weak generalizations in the case where a $p$-Sylow subgroup of $G$ is abelian, not necessarily cyclic. They are partially worked out and will not be included at this time. Their proofs are based on a version of the proof of Theorem 3.1 that does not use the auxiliary cover.

## 3.2 Further properties

We maintain the assumptions of §2.5, along with the assumption that a $p$-Sylow subgroup of $G$ is cyclic of order $p^n$. As this section progresses, we will add more assumptions.

**Lemma 3.6.** *If $b$ indexes an inseparable tail $\overline{X}_b$, then all $\sigma_{i,b}$'s (page 60) are integers.*

*Proof.* Consider the stable model of the strong auxiliary cover $f^{str}$. We know that any branch point of this cover has either prime-to-$p$ or $p$-power order. By Proposition 2.17, any branch point on the generic fiber which specializes to a point in $\overline{X}_b$ must be of $p$-power ramification index. In particular, the ramification index will be $p^d$, where $\overline{X}_b$ is a $p^d$-component. Consider an irreducible component $\overline{V}_b$ of $\overline{Y}^{str}$ lying above $\overline{X}_b$. Then $\overline{V}_b/(\mathbb{Z}/p^d) \to \overline{X}_b$ is totally ramified above the point of intersection $x$ of $\overline{X}_b$ and the rest of $\overline{X}$, and étale elsewhere. If we quotient out by the (normal) $p$-Sylow subgroup of the inertia group $I$ of $\overline{V}_b/(\mathbb{Z}/p^d) \to \overline{X}_b$ above $x$, we obtain a tame cover branched at no more than one point, which must be trivial. Thus $I$ is a $p$-group, and the upper jumps corresponding to the ramification of $\overline{V}_b/(\mathbb{Z}/p^d) \to \overline{X}_b$ are integral by the Hasse-Arf Theorem ([Ser79, V, Theorem 1]). Hence each $\sigma_{i,b}$ is integral by Remark 3.3. □

**Lemma 3.7.** *Let $\overline{x}$ be a singular point of $\overline{X}$ such that there are no étale tails $\overline{X}_b$ with $\overline{x} \prec \overline{X}_b$. Then for any deformation datum above an irreducible component containing $\overline{x}$, the invariant $\sigma_{\overline{x}}$ is an integer.*

*Proof.* Take the strong auxiliary cover $f^{str} : Y^{str} \to X$ of $f$. By Proposition 2.36, this does not change the invariant $\sigma_{\overline{x}}$. Now, $f^{str}$ has Galois group $\mathbb{Z}/p^{\nu} \rtimes \mathbb{Z}/m$, for some

$\nu \leq n$. By construction, $f^{str}$ has no branch points of prime-to-$p$ branching index which specialize outward from $\overline{x}$.

Any irreducible component $\overline{V}$ of $\overline{Y}^{str}$ has decomposition group $\mathbb{Z}/p^r \rtimes \mathbb{Z}/m'$, where $r \leq \nu$ and $m'|m$. If $\overline{W}$ is the component of $\overline{X}$ below $\overline{V}$, then it is impossible for $\overline{W}$ to have exactly one point where the branching index is not a $p$-power (if this were the case, we could quotient out by the subgroup of order $p^r$ to obtain a tamely ramified cover of $\overline{W}$ branched at one point, which is impossible). This means that at intersection points of inseparable tails lying outward from $\overline{x}$ with the rest of $\overline{X}$, the branching index is a power of $p$ (as the intersection point is the only possible branch point of the inseparable tail). By inward induction, the branching index of $\overline{x}$ is a $p$-power. By the definition of $\sigma_{\overline{x}}$, we have $\sigma_{\overline{x}} \in \mathbb{Z}$. □

**Lemma 3.8.** *(i) A new tail $\overline{X}_b$ (or even an inseparable tail which does not contain the specialization of any branch point) has $\sigma_b \geq 1 + 1/m$.*

*(ii) If $\overline{X}_c$ is any $p^d$-tail to which no branch point of $f$ specializes and that borders a $p^e$-component, then $\sigma_b \geq p^{e-d-1}$.*

*(iii) If $\overline{X}_b$ is a primitive tail that borders a $p^e$-component, then $\sigma_b \geq p^{e-1}/m$.*

*Proof.* If we assume that (i) is false, [Ray99, Lemme 1.1.6] shows that each irreducible component above $\overline{X}_b$ is a genus zero Artin-Schreier cover of $\overline{X}_b$. Since no ramification points specialize to these components, this contradicts the three-point condition of the stable model.

For (ii) and (iii) we cite [Pri06, Lemma 19], which shows that $\sigma_{i+1,b} \geq p\sigma_{i,b}$ for all

$i$ where the statement makes sense. Recall that $\sigma_b = \sigma_{e-d,b}$. Now, if $\overline{X}_b$ is inseparable, then $\sigma_b \in \mathbb{Z}$ (by Lemma 3.6). If $\sigma_b = 1$, then the components above the tail are genus zero by [Ray99, Lemme 1.1.6], and they violate the three-point condition of the stable model. So $\sigma_b \geq 2$ for these inseparable tails, and (ii) is proved. Also, $\sigma_{1,b} \geq 1/m$ for $\overline{X}_b$ primitive. Then (iii) follows. □

**Corollary 3.9.** *Let $x$ be a branch point of $f$ with branching index exactly divisible by $p^r$. Suppose that $x$ specializes to $\overline{x}$, which lies on an irreducible component $\overline{W}$ of $\overline{X}$ such that $\overline{f}$ is monotonic from $\overline{W}$. Then either $\overline{W}$ is the original component or $\overline{W}$ intersects a $p^{r+j}$-component, for some $j > 0$.*

*Proof.* By Proposition 2.19, $\overline{W}$ is a $p^r$-component. We note that since $\overline{W}$ contains the specialization of a branch point with branching index divisible by $p$, the deformation data above $\overline{W}$ are multiplicative (and thus identical, by Proposition 2.41). Now, assume that $\overline{W}$ is not the original component. Let $S$ be the set of singular points of $\overline{X}$ lying on $\overline{W}$, and let $s_0 \in S$ be the unique point of $S$ such that $s_0 \prec \overline{W}$. For each $s \in S' := S \backslash \{s_0\}$, we consider $\sigma_s^{\text{eff}}$. Let $B_s$ index the set of all étale tails $\overline{X}_b$ such that $s \prec \overline{X}_b$. Since $x$ specializes to $\overline{W}$, no branch point of $f$ can specialize to any component outward from $\overline{W}$. In particular, each $\overline{X}_b$ is a new tail. Because $\overline{f}$ is monotonic from $\overline{W}$, we can apply Lemma 2.47 to show that

$$\sigma_s^{\text{eff}} - 1 = \sum_{b \in B_s} (\sigma_b - 1).$$

By Lemma 3.8, each $\sigma_b$ in the above sum is greater than 1, so $\sigma_s^{\text{eff}} \geq 1$.

Since the deformation data above $\overline{W}$ are all the same, this shows that each deformation datum $\omega$ must have $\sigma_s \geq 1$ for all $s \in S'$. For a contradiction, assume that $\overline{W}$ does not intersect any $p^{r+j}$-components with $j > 0$. Then any irreducible component $\overline{V}$ lying above $\overline{W}$ is tamely ramified, and we can apply the local vanishing cycles formula (2.7.1). Since the only critical points of $\omega$ lie in $S \cup \{\overline{x}\}$, and $\sigma_{\overline{x}} = 0$, we see that $\sigma_{s_0} \leq 0$. But $\sigma_{s_0} = 0$ contradicts Proposition 2.40, and $\sigma_{s_0} < 0$ is impossible because $\omega$ is multiplicative. Thus we have our contradiction, and the corollary is proved. □

We now give some sufficient criteria for the stable reduction of $f$ to be monotonic. In particular, they are satisfied in all of the cases in Theorem 1.4.

**Proposition 3.10.** *(i) If $G$ is $p$-solvable, then $\overline{f}$ is monotonic.*

*(ii) If $\overline{T}$ is a component of $\overline{X}$ such that there are no new étale tails $\overline{X}_b \succ \overline{T}$, then $\overline{f}$ is monotonic from $\overline{T}$.*

*(iii) If $f$ is a three-point cover of $\mathbb{P}^1$, and $m = 2$, then $\overline{f}$ is monotonic.*

*Proof.* In cases (i) and (iii), assume for a contradiction that $\overline{W}$ is a maximal component of $\overline{X}$ for $\prec$ such that $\overline{f}$ is not monotonic from $\overline{W}$. Suppose that $\overline{W}$ is a $p^i$-component. Let $\overline{W}'$ be a $p^j$-component intersecting $\overline{W}$ such that $\overline{W} \prec \overline{W}'$ and $j > i$. Let $\{w\} = \overline{W}' \cap \overline{W}$.

*To (i):* By Corollary 2.4, we know that there is a prime-to-$p$ group $N$ such that $G/N \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. Since quotienting out by a prime-to-$p$ group does not affect monotonicity, we may assume that $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. Consider the morphism $X^{st} \to X'$ that is "the identity" on $X_{K^{st}}$ and contracts all components of $\overline{X}$ outward from and including $\overline{W}'$. If $Y'$ is the normalization of $X'$ in $K^{st}(Y)$, then $Y'$ is obtained from $Y^{st}$ by contracting all of the components of the special fiber above those components contracted by $X^{st} \to X'$. Let $y$ be a point of $Y'$ lying over the image of $w$ in $X'$ (which we call $w$, by abuse of notation), and consider the map of complete germs $\hat{Y}'_y \to \hat{X}'_w$. This map is Galois with Galois group $G' \cong \mathbb{Z}/p^j \rtimes H$, where $|H|$ is prime-to-$p$, by Proposition 2.17 and the fact that $\overline{f}$ is monotonic from $\overline{W}'$. Let $\hat{V}_v$ be the quotient of $\hat{Y}'_y$ by the subgroup of $G$ of order $p^i$. Then $\phi : \hat{V}_v \to \hat{X}'_w$ is Galois with Galois group $\mathbb{Z}/p^{j-i} \rtimes H$. Note that $w$ is a *smooth* point of $X'$, as we have contracted only a tree of projective lines (of course, $y$ may be quite singular, but it is still a normal point of $Y'$).

Now, $\phi$ is totally ramified above the point $w$, but it is unramified above the height 1 prime $(\pi)$, where $\pi$ is a uniformizer of $R$, because we have quotiented out the generic inertia of $\overline{W}$. Using purity of the branch locus ([Sza09, Theorem 5.2.13]), we see that $\phi$ must be ramified over some height 1 prime $(t)$ such that the scheme cut out by $t$ intersects the generic fiber. Since we have been assuming from the beginning that the branch points of $Y_K \to X_K$ do not collide on the special fiber $\overline{X}_0$, and we have not contracted $\overline{X}_0$, there is at most one branch point on the generic fiber that can specialize to $w$, and thus $(t)$ cuts the generic fiber in exactly one point, and it is the only height 1 prime above which $\phi$ is ramified. So $\phi$ is étale outside of the scheme cut out by $(t)$. We are now in the situation of [Ray94, Lemme 6.3.2], and we conclude that the ramification index at $(t)$ is prime-to-$p$. But this contradicts the fact that the ramification index above $w$ is divisible by $p^{j-i}$.

*To (ii):* We take the auxiliary cover $f^{aux}$ of $f$. Since there are no new étale tails lying outward from $\overline{T}$, the construction of $f^{aux}$ does not introduce any branch points beyond

$\overline{T}$. Also, the Galois group $G^{aux}$ of $f^{aux}$ is $p$-solvable (§2.6). So the proof of part (i) carries through exactly to show that $f^{aux}$ has monotonic stable reduction from $\overline{T}$. Since the construction of $f^{aux}$ does not alter any generic inertia groups of components of $\overline{X}$, $f^{aux}$ has monotonic stable reduction from $\overline{T}$ if and only if $f$ does. So the stable reduction of $f$ is monotonic from $\overline{T}$.

*To (iii):* Let $\Sigma$ be the largest set of $p^j$-components of $\overline{X}$ such that $\Sigma$ contains $\overline{W}'$ and the union $\overline{U}$ of the components in $\Sigma$ is connected. Let $S$ be the set of all singular points of $\overline{X}$ that lie at the intersection of a component in $\Sigma$ with a component not in $\Sigma$. By assumption, $\overline{W} \notin \Sigma$. So $w \in S$. Also, since $\overline{f}$ is assumed to be monotonic from $\overline{W}'$, every $s \in S$ lies on a $p^{j-a_s}$-component, with $a_s > 0$. Let $\overline{X}_s \in \Sigma$ contain $s$, and let $\overline{V}_s$ be an irreducible component of $\overline{Y}$ lying above $\overline{X}_s$. By Proposition 2.42, the bottom differential form $\omega$ for $\overline{V}_s$ has positive invariant $\sigma_s$. Furthermore, $(\overline{V}_s, \omega)$ is a deformation datum of type $(D_{\overline{V}_s}/I_{\overline{V}_s}, \chi)$, where $D_{\overline{V}_s}$ is the decomposition group of the component $\overline{V}_s$ and $I_{\overline{V}_s}$ is the inertia group. Since $m = 2$, and $\chi$ comes from the character of $D_{\overline{V}_s}/I_{\overline{V}_s}$ corresponding to the conjugation action on $I_{\overline{V}_s}$ (Construction 2.39), we have that $(|D_{\overline{V}_s}/I_{\overline{V}_s}|/|D_{\overline{V}_s}/I_{\overline{V}_s} \cap \ker(\chi)|) \,|\, 2$. Since $\overline{V}_s \to \overline{X}_s$ is tamely ramified everywhere, in particular above $s$, Proposition 2.37 shows that $\sigma_s \in \frac{1}{2}\mathbb{Z}$. It follows that $\sigma_s \geq \frac{1}{2}$.

Note that, by Corollary 3.9, no branch point of $f$ with ramification index divisible by $p$ can specialize to any component in $\Sigma$. Then, by repeated application of the local vanishing cycles formula (2.7.1), we obtain the equation

$$\sum_{s\in S}(\sigma_s - 1) = -2.$$

We know that each term in this sum is at least $-\frac{1}{2}$. We claim that at most two of the terms can be non-integral. This will contradict the equation.

We show the claim. For $s \in S$, we know by Lemma 3.7 that $\sigma_s \in \mathbb{Z}$ unless there is an étale tail $\overline{X}_b$ such that $\overline{X}_s \prec \overline{X}_b$. But since $m = 2$, the vanishing cycles formula (3.1.2), in conjunction with with Lemma 2.26, shows that there can be at most two étale tails. So at most two elements $s$ of $S$ can have non-integral $\sigma_s$. This proves the claim, and we obtain the desired contradiction. □

For the rest of this section, assume that $f : Y \to X$ is a three-point cover of $\mathbb{P}^1$ with bad reduction.

**Proposition 3.11.** *(i) Suppose $\overline{f}$ is monotonic. The stable reduction $\overline{X}$ has at most one layer of $p$-components, i.e., no two $p$-components intersect each other.*

*(ii) If $G$ is $p$-solvable with $m > 1$, then there is at most one layer of $p^i$-components for each $0 \leq i \leq n$.*

*Proof. To (i)*: Suppose, for a contradiction, that $\overline{X}$ has a maximal chain of intersecting $p$-components of length greater than one, the maximal component of the chain being $\overline{W}$. Then, by Proposition 2.24, $\overline{W}$ is not a tail, and monotonicity shows that any component lying outward from $\overline{W}$ is étale (and a tail, by Proposition 2.22). Consider a deformation datum $(\overline{V}, \omega)$ corresponding to some irreducible component $\overline{V}$ of $\overline{Y}$ lying above $\overline{W}$. By

Proposition 2.40, the only possibilities for critical points of $\omega$ are points of intersection of $\overline{W}$ with étale tails, the point of intersection of $\overline{W}$ with the immediately preceding irreducible component of $\overline{X}$, or the specialization of a wild branch point of $f$ to $\overline{X}$. Now, for $x_b$ the intersection of $\overline{W}$ with an étale tail $\overline{X}_b$, the vanishing cycles formula (3.1.2) shows that $\sigma_{x_b}$ is an integer if and only if $\overline{X}_b$ is the *only* étale tail of $\overline{X}$. If this is the case, then $\overline{W}$ intersects exactly two components of $\overline{X}$, and $\overline{V}$ is of genus zero, as it is a degree $p$ inseparable extension of a tame cyclic cover of $\overline{X}$ branched at two points. Then $\overline{V}$ is totally ramified above these two points, and the three-point condition of the stable model is violated unless a wild branch point specializes to some $w \in \overline{W}$. But this violates Corollary 3.9.

Now assume that there is more than one étale tail of $\overline{X}$. Let $B_W$ index the set of étale tails $\overline{X}_b$ intersecting $\overline{W}$, and let $C$ be the set of all critical points of $\omega$. For each $b \in B_W$, let $x_b$ be the intersection of $\overline{W}$ with $\overline{X}_b$. The vanishing cycles formula (3.1.2), along with Lemma 3.8, shows that $0 < \sigma_{x_b} < 1$ for $\overline{X}_b$ primitive and $1 < \sigma_{x_b} < 2$ for $\overline{X}_b$ new. Also, $\sum_{b \in B_W} \langle \sigma_{x_b} \rangle \leq 1$. Since $\sum_{c \in C}(\sigma_c - 1) = -2$, we have $\sum_{c \in C} \sigma_c \in \mathbb{Z}$. Let $w$ be the point of intersection of $\overline{W}$ and the interior part of $\overline{X}$. Then $w$ is the only element of $C \backslash B_W$ (no wild branch point specializes to $\overline{W}$ by Corollary 3.9). Thus $\sum_{c \in C} \langle \sigma_c \rangle = 1$. This means that $\sum_{c \in C}(\lfloor \sigma_c \rfloor - 1) = -3$. Since $\lfloor \sigma_b \rfloor - 1$ is $-1$ for a primitive tail $\overline{X}_b$ and 0 for a new tail $\overline{X}_b$, then $\lfloor \sigma_w \rfloor$ equals $-1$ if there is a branch point specializing outward from $w$ or $-2$ if there is no such branch point. We are now in the situation of [Wew03a, Lemma 2.8], which shows that the deformation datum on $\overline{V}$ is not additive. Since $\sigma_w < 0$, the deformation datum has multiple poles above $w$, so it is not multiplicative either. This contradiction finishes the proof of (i). Notice that this contradiction did not depend on the three-point condition for the stable model.

*To (ii)*: Since $G$ is $p$-solvable, $\overline{f}$ is monotonic by Proposition 3.10. The case $i = 0$ follows immediately from Proposition 2.22. Since $G$ is $p$-solvable, there is a prime-to-$p$ normal subgroup $H \leq G$ such that $G/H \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, where the action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful (Corollary 2.4). Let $h : Z \to X$ be the corresponding cover with Galois group $G/H$. Since $m > 1$, $G/H$ is not cyclic, and thus $h$ is branched at three points. In addition, at least two of those points must be branched of prime-to-$p$-order, because otherwise $h$ would have a quotient which is a $\mathbb{Z}/m$-cover branched at fewer than two points (using that all branching indices are prime to $p$ or powers of $p$). Thus the stable reduction $\overline{h}$ has at least two primitive tails.

Let $f^{st} : Y^{st} \to X^{st}$ be the stable model of $f$, with $\overline{f} : \overline{Y} \to \overline{X}$ the stable reduction. Say there are two intersecting $p^i$-components of $\overline{X}$, for some $i > 0$.

Consider the unique subgroup $H' \subseteq G$ which contains $H$ such that $H'/H \cong \mathbb{Z}/p^{i-1}$. Then $f^{st}/H'$ is a semistable model of a three-point cover with at least two primitive tails that contains two intersecting $p$-components. In this case, since the contradiction obtained in part (i) does not rely on the three-point condition of the stable model, so it gives us a contradiction here. □

**Lemma 3.12.** *If $G$ is $p$-solvable with $m > 1$, then the original component is a $p^n$-component, and all differential forms above the original component are multiplicative.*

*Proof.* Since $G$ is $p$-solvable, we know by Corollary 2.4 that $f : Y \to X$ has a quotient cover $Y' \to X$ with Galois group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, and thus a quotient cover $h : Z \to X$ with Galois group $\mathbb{Z}/p \rtimes \mathbb{Z}/m$. If all branch points of $h$ are of prime-to-$p$ ramification index, then [Wew03a, §1.4] shows that $h$ is of *multiplicative type* in the language of [Wew03a]. Then $h$ has bad reduction by [Wew03a, Corollary 1.5], and the original component for the stable reduction $\overline{Z} \to \overline{X}$ is a $p$-component. Furthermore, the deformation datum on the irreducible component of $\overline{Z}$ above the original component of $\overline{X}$ is multiplicative (also due to [Wew03a, Corollary 1.5]).

If $h$ has a branch point $x$ with ramification index divisible by $p$, then $h$ has bad reduction (if $h$ had good reduction $\overline{h}$, then $\overline{h}$ would be generically étale, but not tamely ramified–this cannot happen unless branch points collide on the special fiber). By Proposition 2.19, $x$ specializes to a $p$-component. By Proposition 3.11, this is the original component $\overline{X}_0$, which is the only $p$-component. The deformation datum above $\overline{X}_0$ must be multiplicative here, as it contains the specialization of a branch point with $p$ dividing the branching index (see after Proposition 2.40).

So in all cases, the original component is a $p$-component for $h$ with multiplicative deformation datum. Thus the "bottom" differential form above $\overline{X}_0$ for $f$ is multiplicative. Now, consider the $\mathbb{Z}/p^n$-cover $Y' \to Z' := Y'/(\mathbb{Z}/p^n)$. If $Y'$ has stable model $(Y')^{st}$ and stable reduction $\overline{Y}'$, let $(Z')^{st} := (Y')^{st}/(\mathbb{Z}/p^n)$. Then there is a canonical map $(Z')^{st} \to X^{st}$. Say $\overline{V}'$ is an irreducible component of the special fiber of $(\overline{Z}')^{st}$ lying above $\overline{X}_0$, let $\eta$ be its generic point, and consider the ring $C := \hat{\mathcal{O}}_{(Z')^{st},\eta}$. The normalization of Spec $C$ in $Y'$ is given, after a possible extension of $K^{st}$, by an equation $y^{p^n} = u$. Since the bottom differential form above $\overline{X}_0$ for $f$ is multiplicative, the reduction $\overline{u}$ of $u$ is not a $p$th power in the residue field of $C$. This means that there exists only one point of $\overline{Y}'$ above $\eta$, and thus by monotonicity, $\overline{X}_0$ is a $p^n$-component. Finally, Proposition 2.41 shows that all the differential forms above $\overline{X}_0$ for $f$ are multiplicative. □

**Proposition 3.13.** *(i) If $G$ is $p$-solvable with $m > 1$, then there are no inseparable tails.*

*(ii) If $G$ is $p$-solvable with $m > 1$, then there are no new tails.*

*Proof. To (i):* Say there is an inseparable tail $\overline{X}_b$ that is a $p^i$-component with ramification invariant $\sigma_b$. By Lemma 3.6, $\sigma_b$ is an integer. By Lemma 3.8, $\sigma_b > 1$ if $\overline{X}_b$ does not contain the specialization of any branch point. Assume for the moment that this is the case. Then $\sigma_b \geq 2$. We let $f' : Y/H \to X$ be the unique quotient cover of $f : Y \to X$ with Galois group $\mathbb{Z}/p^{n-i} \rtimes \mathbb{Z}/m$. We know $f'$ is branched at three points, with at least two having prime-to-$p$ ramification index. Thus the stable reduction $\overline{f}'$ has at least two primitive tails. It also has a new tail corresponding to the image of $\overline{X}_b$, which has ramification invariant $\sigma_b \geq 2$. Then the right-hand side of (3.1.2) for the cover $f'$ is greater than 1. The left-hand side equals 1, so we have a contradiction.

We now prove that no branch point of $f$ specializes to $\overline{X}_b$. Such a branch point $x$ would have ramification index $p^i s$, where $p \nmid s$.

Let $H'$ be such that $G/H' \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ (Corollary 2.4). Then in $Y/H' \to X$, $x$ would have ramification index $p^i$. So in $f' : Z = Y/H \to X$, $x$ would have ramification index 1. Thus $Z \to X$ would be branched in at most two points, which contradicts the fact that $f'$ is not cyclic.

*To (ii):* Suppose there is a new tail $\overline{X}_b$ with ramification invariant $\sigma_b$. If $\sigma_b \in \mathbb{Z}$, we get the same contradiction as in the inseparable case. If $\sigma_b \notin \mathbb{Z}$, then $\overline{X}_b$ is branched in only one point with inertia group $\mathbb{Z}/p^i \rtimes \mathbb{Z}/m_b$ for $i \geq 1$, $m_b > 1$. Let $\overline{Y}_b$ be an irreducible component of $\overline{Y}$ lying above $\overline{X}_b$. If $P$ is a $p$-Sylow subgroup of $G$, consider the quotient cover $\overline{Y}/P \to \overline{X}$. The image of $\overline{Y}_b$ in $\overline{Y}/P$ is a cover of $\overline{X}_b$ branched in one point of ramification index $m_b > 1$. This is a contradiction. □

# Chapter 4

# Proof of the Main Theorem

In this chapter, we will prove Theorem 1.4. Let $f : Y \to X = \mathbb{P}^1$ be a three-point Galois cover defined over $\overline{\mathbb{Q}}$. For an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p^{ur}}$, let $f_\iota$ be the base change of $f$ to $\overline{\mathbb{Q}_p^{ur}}$ via $\iota$. The following proposition shows that, for the purposes of Theorem 1.4, we need only consider covers defined over $\overline{\mathbb{Q}_p^{ur}}$.

**Proposition 4.1.** *Let $K_{gl}$ be the field of moduli of $f$ (with respect to $\mathbb{Q}$) and let $K_{loc,\iota}$ be the field of moduli of $f_\iota$ with respect to $\mathbb{Q}_p^{ur}$. Fix $n \geq 0$, and suppose that for all embeddings $\iota$, the $n$th higher ramification groups of $K_{loc,\iota}/\mathbb{Q}_p^{ur}$ for the upper numbering vanish. Then all the $n$th higher ramification groups of $K_{gl}/\mathbb{Q}$ above $p$ for the upper numbering vanish.*

*Proof.* Pick a prime $q$ of $K_{gl}$ above $p$. We will show that the $n$th higher ramification groups at $q$ vanish. Choose a place $r$ of $\overline{\mathbb{Q}}$ above $q$. Then $r$ gives rise to an embedding $\iota_r : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p^{ur}}$ preserving the higher ramification filtrations at $r$ for the upper numbering (and the lower numbering). Specifically, if $K/\mathbb{Q}_p^{ur}$ is a finite extension such that the $n$th higher ramification group for the upper numbering vanishes, then the $n$th higher ramification group for the upper numbering vanishes for $\iota_r^{-1}(K)/\mathbb{Q}$ at the unique prime of $\iota_r^{-1}(K)$ below $r$. By assumption, the $n$th higher ramification group for the upper numbering vanishes for $K_{loc,\iota_r}/\mathbb{Q}_p^{ur}$. So in order to prove the proposition, it suffices to show that the field of moduli $K_{gl}$ of $f$ is contained in $K' := \iota_r^{-1}(K_{loc,\iota_r})$.

Pick $\sigma \in G_{K'}$. Then $\sigma$ extends by continuity to a unique automorphism $\tau$ in $G_{K_{loc,\iota_r}}$. By the definition of a field of moduli, $f_{\iota_r}^{\tau} \cong f_{\iota_r}$. But then $f^{\sigma} \cong f$. By the definition of a field of moduli, $K_{gl} \subseteq K'$. □

So, in order to prove Theorem 1.4, we can consider three-point covers defined over $\overline{\mathbb{Q}_p^{ur}}$. In fact, we generalize slightly, and consider three-point covers defined over algebraic closures of complete mixed characteristic discrete valuation fields with algebraically closed residue fields. In particular, throughout this chapter, $K_0$ is the fraction field of the ring $R_0$ of Witt vectors over an algebraically closed field $k$ of characteristic $p$. Also write $K_n := K_0(\zeta_{p^n})$, with valuation ring $R_n$. Let $G$ be a finite group with a cyclic $p$-Sylow subgroup of order $p^n$, and $m = |N_G(P)/Z_G(P)|$. We assume $f : Y \to X = \mathbb{P}^1$ is a three-point $G$-Galois cover of curves, *a priori* defined over some finite extension $K/K_0$. Since $K$ has cohomological dmension 1, the field of moduli of $f$ relative to $K_0$ is the same as the minimal field of definition of $f$ that is an extension of $K_0$ ([CH85, Prop. 2.5]). We will therefore go back and forth between fields of moduli and fields of definition without futher notice.

In the next three sections, we deal separately with the case of a $p$-solvable group $G$ with $m > 1$, the case of a group $G$ with $m = 1$, and the case of a group $G$ with $m = 2$. In most cases, we in fact determine more than we need for Theorem 1.4; namely, we determine bounds on the higher ramification filtrations of the extension $K^{st}/K_0$, where $K^{st}$ is the minimal field of definition of the stable model of $f$. Our default smooth model $X_R$ of $X$ is always the unique one so that the specializations of 0, 1, and $\infty$ do not collide on the special fiber.

## 4.1 The case where $G$ is $p$-solvable and $m > 1$

We know by Corollary 2.4 that $G$ has a quotient of the form $G/N \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, where the action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful. So we begin by considering only covers where $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, the action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful, and $m > 1$. Our first aim will be to show that such a cover is defined as a mere cover over $K_0$. We will deal with the general $p$-solvable case for $m > 1$ afterwards.

### 4.1.1 The case of a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover

Say $f : Y \to X$ is a three-point $G$-cover with $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ and the conjugation action of $\mathbb{Z}/m$ is faithful. Now, there is an intermediate $\mathbb{Z}/m$-cover $h : Z \to X$ where $Z = Y/(\mathbb{Z}/p^n)$. If $g : Y \to Y/N$ is the quotient map, then $f = h \circ g$. Because it will be easier for our purposes here, let us assume that the three branch points of $f$ are at $x_1, x_2$, and $x_3$ which are elements of $R_0$, none of which has the same reduction to $k$ (in particular, none is $\infty$). Since the $m$th roots of unity are contained in $K_0$, the cover $h$ can be given birationally by the equation $z^m = (x - x_1)^{a_1}(x - x_2)^{a_2}(x - x_3)^{a_3}$ with $0 \leq a_i < m$ for all $i \in \{1, 2, 3\}$, where $a_1 + a_2 + a_3 \equiv 0 \pmod{m}$. Fix a primitive $m$th root of unity $\zeta_m$, and a generator $g \in \mathbb{Z}/m$ such that $g^*z = \zeta_m z$. Let $\overline{f} : \overline{Y} \to \overline{X}$ be the stable reduction of $f$, relative to the standard smooth model $X_R = \mathbb{P}^1_R$ for some finite extension $R/R_0$, and let $\overline{X}_0$ be the original component of $\overline{X}$. We know from Lemma 3.12 that $\overline{X}_0$ is a $p^n$-component, and all of the deformation data above $\overline{X}_0$ are multiplicative.

Consider the $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-cover $f' : Y' \to X$, where $Y'$ is the quotient of $Y$ by the unique subgroup of order $p^{n-1}$ of $G$. The stable reduction $\overline{f}' : \overline{Y}' \to \overline{X}'$ of this cover has a multiplicative deformation datum $(\omega, \chi)$ over the original component $\overline{X}_0$, with $\chi : \mathbb{Z}/m \hookrightarrow \mathbb{F}_p^\times$ such that $g^*(\omega) = \chi(1)\omega$. For all $x \in \overline{X}_0$, recall that $(h_x, m_x)$ is the signature of the deformation datum at $x$, and $\sigma_x = h_x/m_x$ (see §2.7.2). Also, since there are no new tails (Proposition 3.13), Proposition 3.11 shows that the stable reduction $\overline{X}'$ consists only of the original component $\overline{X}_0$ along with a primitive étale tail $\overline{X}_i$ for each branch point $x_i$ of $f$ (or $f'$) with prime-to-$p$ ramification index. The tail $\overline{X}_i$ intersects $\overline{X}_0$ at the specialization of $x_i$ to $\overline{X}_0$.

**Proposition 4.2.** *For $i = 1, 2, 3$, let $\overline{x}_i$ be the specialization of $x_i$ to $\overline{X}_0$. For short, write $h_i$, $m_i$, and $\sigma_i$ for $h_{\overline{x}_i}$, $m_{\overline{x}_i}$, and $\sigma_{\overline{x}_i}$.*

*(i) For $i = 1, 2, 3$, $h_i \equiv a_i/\gcd(m, a_i) \pmod{m_i}$.*

*(ii) In fact, the $h_i$ depend only on the $\mathbb{Z}/m$-cover $\eta : Z \to X$.*

*Proof. To (i):* (cf. [Wew03a, Proposition 2.5]) Let $\overline{Z}_0$ be the unique irreducible component lying above $\overline{X}_0$, and suppose that $\overline{z}_i \in \overline{Z}_0$ lies above $\overline{x}_i$. If $t_i$ is a formal parameter at $\overline{z}_i$, we have

$$\omega = (c_0 t_i^{h_i - 1} + \sum_{j=1}^{\infty} c_j t_i^{h_i - 1 + j}) dt_i$$

in a formal neighborhood of $\overline{z}_i$. Now, $g^*\omega = \chi(1)\omega$. Thus $(g^{a_i})^*\omega = \chi(a_i)\omega$. Also, a local calculation (using $z^m = (\text{unit})(x - x_i)^{a_i}$) shows that $(g^{a_i})^* t_i = \chi(\gcd(m, a_i)) t_i$. Thus $(g^{a_i})^*\omega = \chi(h_i \gcd(m, a_i))\omega$. Since $\chi(1)$ has order $m$, we conclude that

$$h_i \gcd(m, a_i) \equiv a_i \pmod{m}. \tag{4.1.1}$$

It is clear that the ramification index $m_i$ at $\overline{x}_i$ is $m/\gcd(m, a_i)$. Dividing (4.1.1) by $\gcd(m, a_i)$ yields (i).

*To (ii):* Since we know the congruence class of $h_i$ modulo $m_i$, it follows that the fractional part $\langle \sigma_i \rangle$ of $\sigma_i$ is determined by $\eta : Z \to X$. But if $x_i$ corresponds to a primitive tail, we know that $0 < \sigma_i < 1$. If $x_i$ corresponds to a wild branch point, then $\sigma_i = 0$. Thus $\sigma_i$ is determined by $\langle \sigma_i \rangle$, so it is determined by $h : Z \to X$. Since $h_i = \sigma_i m_i$, we are done. □

**Corollary 4.3.** *The differential form $\omega$ corresponding to the cover $f' : Y' \to X$ is determined (up to multiplication by a scalar) by $h : Z \to X$.*

*Proof.* Proposition 4.2 determines the divisor corresponding to $\omega$ from $h : Z \to X$. Two meromorphic differential forms on a complete curve can have the same divisor only if they differ by a scalar multiple. □

We will now show that $h : Z \to X$ determines not only the differential form $\omega$, but actually the entire cover $f : Y \to X$ as a mere cover. We will do this in several stages, using an induction.

**Proposition 4.4.** *Assume $m > 1$.*

*(i) If $f : Y \to X$ is a three-point $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover (with faithful conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$) defined over some finite extension $K/K_0$, then it is determined as a mere cover by the map $\eta : Z = Y/(\mathbb{Z}/p^n) \to X$.*

*(ii) If $f : Y \to X$ is a three-point $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover (with faithful conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$) defined over some finite extension $K/K_0$, its field of moduli (as a mere cover) with respect to $K_0$ is $K_0$, and $f$ can be defined over $K_0$ (as a mere cover).*

*(iii) In the situation of part (ii), the field of moduli of $f$ (as a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover) with respect to $K_0$ is $K_n = K_0(\zeta_{p^n})$. Thus $f$ is defined over $K_n$ (as a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover).*

*Proof. To (i):* We first prove the case $n = 1$. Write $(f')^{st} : (Y')^{st} \to (X')^{st}$ for the stable model of $f'$, let $\overline{f}' : \overline{Y}' \to \overline{X}'$ be the stable reduction of $f'$, and let $\overline{X}_0$ be the original component of $\overline{X}'$. Write $(Z')^{st}$ for $(Y')^{st}/(\mathbb{Z}/p)$ and $\overline{Z}'$ for $\overline{Y}'/(\mathbb{Z}/p)$. We know from Corollary 4.3 that $\eta$ determines (up to a scalar multiple) the logarithmic differential form $\omega$ that is part of the deformation datum $(\overline{Z}_0, \omega)$ on the irreducible component $\overline{Z}_0$ above $\overline{X}_0$. Let $\xi$ be the generic point of $\overline{Z}_0$. Then $\omega$ is of the form $d\overline{u}/\overline{u}$, where $\overline{u} \in k(\overline{Z}_0)$ is the reduction of some function $u \in \hat{\mathcal{O}}_{(Z')^{st},\xi}$. Moreover, we can choose $u$ such that the cover $Y' \to Z'$ is given birationally by extracting a $p$th root of $u$ (viewing $u \in K(Z) \cap \hat{\mathcal{O}}_{(Z')^{st},\xi}$). That is, $K(Y') = K(Z)[t]/(t^p - u)$. We wish to show that knowledge of $d\overline{u}/\overline{u}$ up to a scalar multiple $c \in \mathbb{F}_p^{\times}$ determines $u$ up to raising to the $c$th power, and then possibly multiplication by a $p$th power in $K(Z)$ (as this shows $Y' \to X$ is uniquely determined as a mere cover). This is clearly equivalent to showing that knowledge of $d\overline{u}/\overline{u}$ determines

$u$ up to a $p$th power.

Let us say that there exists $u, v \in K(Z) \cap \mathcal{O}_{(Z')^{st},\xi}$ such that $d\overline{u}/\overline{u} = d\overline{v}/\overline{v}$. Then $\overline{u} = \overline{\kappa v}$, with $\overline{\kappa} \in k$. Since $k$ is algebraically closed, $\overline{\kappa}$ is a $p$th power, and thus lifts to some $p$th power $\kappa$ in $K$. Multiplying $v$ by $\kappa$, we can assume that $\overline{u} = \overline{v}$. Consider the cover $Y'' \to Z$ given birationally by the field extension $K(Y'') = K(Z)[t]/(t^p - u/v)$. Since $\overline{u} = \overline{v}$, we have that $u/v$ is congruent to 1 in $\mathcal{O}_{(Z')^{st},\xi}$. This means that the cover $Y'' \to Z$ cannot have multiplicative reduction (see Remark 2.33 (i)). But the cover $Y'' \to Z \to X$ is a $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-cover, branched at at most three points, so it must have multiplicative reduction if it is nontrivial (Lemma 3.12). Thus it is trivial, which means that $u/v$ is a $p$th power in $K(Z)$, i.e., $u = \phi^p v$ for some $\phi \in K(Z)$. This proves the case $n = 1$.

For $n > 1$, we proceed by induction. We assume that (i) is known for $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$-covers. Given $\eta : Z \to X$, we wish to determine $u \in K(Z)^\times/(K(Z)^\times)^{p^n}$ such that $K(Y)$ is given by $K(Z)[t]/(t^{p^n} - u)$. By the induction hypothesis, we know that $u$ is well-determined up to multiplication by a $p^{n-1}$st power. Suppose that extracting $p^n$th roots of $u$ and $v$ both give $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-covers branched at $0, 1$, and $\infty$. Consider the cover $Y'' \to Z \to X$ given birationally by $K(Y'') = K(Z)[t]/(t^{p^n} - u/v)$. Since $u/v$ is a $p^{n-1}$st power in $K(Z)$, this cover splits into a disjoint union of $p^{n-1}$ different $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-covers. By part (i), each of these covers is given by extracting a $p$th root of some power of $u$ itself! So $\sqrt[p^{n-1}]{u/v} = u^c w^p$, where $c \in (\mathbb{Z}/p)^\times$. Thus $v = u^{1-p^{n-1}c} w^{p^n}$, which means that extracting $p^n$th roots of either $u$ or $v$ gives the same mere cover.

*To (ii):* We know that the cyclic cover $h$ of part (i) is defined over $K_0$, because we have written it down explicitly. Now, for $\sigma \in G_{K_0}$, $f^\sigma$ is a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover with quotient cover $h$, branched at 0, 1, and $\infty$. By part (i), there is only one such cover, so $f^\sigma \cong f$ as mere covers. Thus the field of moduli of $f$ as a mere cover with respect to $K_0$ is $K_0$. By [CH85], Proposition 2.5, $K_0$ is also a field of definition.

*To (iii):* Since $f$ is defined over $K_0$ as a mere cover, it is certainly defined over $K_n$ as a mere cover. So we have a (not necessarily $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-equivariant) isomorphism $\phi : f \to f^\sigma$ for all $\sigma \in G_{K_n}$. Let $\alpha$ be the automorphism of $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ such that for all $g \in \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, $\phi\alpha(g) = g\phi$. By Kummer theory, we can write $K_n(Z) \hookrightarrow K_n(Y)$ as a Kummer extension, with Galois action defined over $K_n$. This means that $\alpha(g) = g$ for $g \in \mathbb{Z}/p^n$. Furthermore, $h : Z \to X$ is defined over $K_0$ as a $\mathbb{Z}/m$-cover. Thus, $\alpha(\overline{g}) = \overline{g}$ for all $g$, where $\overline{g}$ represents the reduction of $g$ to $\mathbb{Z}/m$. But the only automorphisms of $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ satisfying both of these properties are inner, so $\alpha(g) = \gamma g \gamma^{-1}$, for some $\gamma$ independent of $g$. Replacing $\phi$ with $\phi\gamma$ gives a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-equivariant automorphism $f \to f^\sigma$, which shows that the field of moduli of $f$ with respect to $K_0$ is $K_n$. Since $K_0$ has cohomological dimension 1, we see that $f : Y \to X$ is defined over $K_n$ as a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover. □

We know from Proposition 4.4 that $f$ is defined over $K_0$ as a mere cover and over $K_n$ as a $G$-cover. Recall from Section 2.5 that the minimal field of definition of the stable model $K^{st}$ is the field cut out by the subgroup $\Gamma^{st} \leq G_{K_0}$ that acts trivially on the stable reduction $\overline{f} : \overline{Y} \to \overline{X}$. Recall also that the action of $G_{K_n}$ centralizes the action of $G$.

**Lemma 4.5.** *If $g \in G_{K_n}$ acts on $\overline{Y}$ with order $p$ and acts on an irreducible component*

$\overline{V}$ *of* $\overline{Y}$*, then* $g$ *acts trivially on* $\overline{V}$*.*

*Proof.* First, note that since each tail $\overline{X}_b$ of $\overline{X}$ is primitive (Proposition 3.13), each contains the specialization of a $K_0$-rational point, and thus $g$ fixes that point along with the point of intersection of $\overline{X}_b$ with the rest of $\overline{X}$. Since the action of $g$ on $\overline{X}$ has order $p$, and there exist no nontrivial automorphisms of $\mathbb{P}^1_k$ with order $p$, $g$ fixes each tail of $\overline{X}$ pointwise. Since $g$ fixes the original component pointwise as well, $g$ fixes all of $\overline{X}$ pointwise by an easy inward induction. So if $g$ acts on $\overline{V}$, it acts "vertically," as an element of $D/I$, where $D \subseteq G$ is the decomposition group of $\overline{V}$ and $I$ is the inertia group. The group $D/I$ is of the form $\mathbb{Z}/p^r \rtimes \mathbb{Z}/m'$ for some $r \leq n$, $1 \neq m'|m$, and $D/I$ has trivial center. But $g$ commutes with $D/I$, as it commutes with $G$. Thus $g$ acts trivially on $\overline{V}$. □

**Lemma 4.6.** *If* $g \in G_{K_n}$ *acts on* $\overline{Y}$ *with order* $p$*, then* $g$ *acts trivially on* $\overline{Y}$*.*

*Proof.* We already know that $g$ acts trivially on $\overline{X}$. Recall Lemma 3.12, which states that the original component $\overline{X}_0$ of $\overline{X}$ is a $p^n$-component. Then $g$ acts trivially above $\overline{X}_0$, because the cardinality of the fiber of $f$ above any point of $\overline{X}_0$ is prime to $p$. We now proceed by outward induction. Suppose $g$ acts trivially above a tree $T$ of components of $\overline{X}$ containing the original component. Let $\overline{W}$ be a component of $\overline{X}$ intersecting $T$ but not lying in $T$. Let $\overline{V}$ be a component of $\overline{Y}$ lying above $\overline{W}$. Then $g$ acts on $\overline{V}$, because it fixes any intersection point of $\overline{V}$ with a component of $\overline{Y}$ lying above $T$. By Lemma 4.5, $g$ acts trivially on $\overline{V}$. This completes the induction. □

The following is the main proposition of §4.1.1:

**Proposition 4.7.** *Assume $m > 1$. Let $f : Y \to X$ be a three-point $G$-cover, where $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ (with faithful conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$). Then if $K^{st}$ is the field of definition of the stable model $f^{st} : Y^{st} \to X^{st}$, we have that $K^{st}/(K^{st} \cap K_n)$ is a tame extension.*

*Proof.* We need only show that no element of $G_{K_n}$ acts with order $p$ on $\overline{Y}$, as this will show that $p \nmid [K^{st} : K_n]$. This follows from Lemma 4.6. □

### 4.1.2 The case of a general $p$-solvable cover, $m > 1$

**Proposition 4.8.** *Let $G$ be a finite $p$-solvable group with a cyclic $p$-Sylow subgroup $P$ of order $p^n$. Assume $m = |N_G(P)/Z_G(P)| > 1$. If $f : Y \to X$ is a three-point $G$-cover of $\mathbb{P}^1$, then the field of moduli $K$ of $f$ relative to $K_0$ is a tame extension of $K_0(\zeta_{p^n})$. Furthermore, if $K^{st}/K_0$ is the minimal extension over which the stable reduction of $f$ is defined, then $K^{st}/(K^{st} \cap K_n)$ is a tame extension. Furthermore, the $n$th higher ramification groups for the upper numbering of the extension $K^{st}/K_0$ vanish.*

*Proof.* By Corollary 2.4, we know that there is a prime-to-$p$ subgroup $N$ such that $G/N$ is of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. Let $f : Y \to X$ be a $G$-cover branched at $0, 1$, and $\infty$, defined over some finite extension of $K_0$, and let $f^\dagger : Y^\dagger \to X$ be the quotient $G/N$-cover. We know from Proposition 4.7 that $f^\dagger$ is defined over $K_n$ as a $G/N$-cover and has stable reduction defined over a tame extension $K'$ of $K_n$. Let $\overline{f} : \overline{Y} \to \overline{X}$ and $\overline{f}^\dagger : \overline{Y}^\dagger \to \overline{X}$ be the stable reductions of $f$ and $f^\dagger$, respectively. The branch points of $Y \to Y^\dagger$ are all ramification points of $f^\dagger$, because $f^\dagger$ is branched at three points. Thus, they do not collide on $\overline{Y}^\dagger$. The usage of $\overline{X}$ for both covers is justified because the stable reduction of $X$ is the same for both covers (in fact, $\overline{Y}^{\dagger} = \overline{Y}/N$). Therefore, $\overline{Y} \to \overline{Y}^{\dagger}$ is a finite prime-to-$p$ map (an $N$-cover). Then Lemma 2.30 shows that the stable model of $Y \to Y^{\dagger}$ is defined over a tame extension $K''$ of $K'$. Lemma 2.31 then shows that the stable model of $f$ is defined over a tame extension $K'''$ of $K''$. Thus the field of moduli of $f$ is contained in $K'''$, which is a tame extension of $K'$, and hence a tame extension of $K_n$.

By [Ser79, Corollary to IV, Proposition 18], the $n$th higher ramification groups for the upper numbering of the extension $K^{st}/K_0$ vanish. □

## 4.2 The case where $m = 1$

Since $N_G(P) = Z_G(P)$, Lemma 2.2 shows that $G$ has a quotient of the form $G/N \cong \mathbb{Z}/p^n$. In particular, $G$ is $p$-solvable. As in §4.1, we start by analyzing three-point $\mathbb{Z}/p^n$-covers. Finding the field of moduli is easy in this case, so we give a complete description of the stable reduction of such covers.

### 4.2.1 The case of a $\mathbb{Z}/p^n$-cover

Let $G = \mathbb{Z}/p^n$. Assume $f : Y \to X$ is a three-point $G$-cover branched at $0, 1$, and $\infty$.

**Proposition 4.9.** *The field of moduli of $f : Y \to X$ relative to $K_0$ is $K_n = K_0(\zeta_{p^n})$.*

*Proof.* Since the field of moduli of $f$ relative to $K_0$ is the intersection of all extensions of $K_0$ which are fields of definition of $f$, it suffices to show that $K_n$ is the minimal such extension. By Kummer theory, $f$ can be defined over $\overline{K_0}$ birationally by the equation $y^{p^n} = x^a(x-1)^b$, for some integral $a$ and $b$. The Galois action is generated by $y \mapsto \zeta_{p^n} y$. This cover is clearly defined over $K_n$ as a $G$-cover.

Since $f$ is connected, $f$ is totally ramified above at least one of the branch points $x$ (i.e., with index $p^n$). Let $y \in Y$ be the unique point above $x$. Assume $f$ is defined over some finite extension $K/K_0$ as a $G$-cover, where $Y$ and $X$ are considered as $K$-varieties. Then, by [Ray99, Proposition 4.2.11], the residue field $K(y)$ of $y$ contains the $p^n$th roots of unity. Since $y$ is totally ramified, $K(y) = K(x) = K$, and thus $K \supseteq K_n$. So $K_n$ is the minimal extension of $K_0$ which is a field of definition of $f$. □

In the rest of this section, we give a full, explicit determination of the stable reduction of three-point $G$-covers $f : Y \to X$. We assume throughout that $p \neq 2$ (this case is more difficult, and will not be included). From §2.2, such a cover is given by a triple $(x, y, (xy)^{-1})$ of elements of $G$ such that $x$ and $y$ generate $G$. Since $G$ is cyclic, we see that at least two elements of the set $\{x, y, (xy)^{-1}\}$ have order $p^n$. This means that $f$ is totally ramified above at least two points. Then $f$ can be given by an equation of the form $y^{p^n} = cx^a(x-1)^b$. If $f$ is totally ramified above $0, 1$, and $\infty$, then $a, b$, and $a+b$ are prime to $p$. If $f$ is totally ramified above two points (without loss of generality 0 and $\infty$) and ramified of index $p^s, s < n$ above 1, then $a$ and $a+b$ are prime to $p$, whereas $v_p(b) = n - s$.

As in §2.5, write $f^{st} : Y^{st} \to X^{st}$ for the stable model of $f$, and $\overline{f} : \overline{Y} \to \overline{X}$ for the stable reduction. Note that $\overline{f}$ is monotonic, by Proposition 3.10.

**Lemma 4.10.** *The stable reduction $\overline{X}$ contains exactly one étale tail $\overline{X}_b$, which is a new tail. Let $d = \frac{a}{a+b}$.*

*(i) If $f$ is totally ramified above $0, 1,$ and $\infty$, then $\overline{X}_b$ corresponds to the disk of radius*

*$|e|$ centered at $d$, where $v(e) = \frac{1}{2}(n + \frac{1}{p-1})$.*

*(ii) If $f$ is totally ramified above* 0 *and* $\infty$*, and ramified of order $p^s$ above* 1*, then $\overline{X}_b$ corresponds to the disk of radius $|e|$ centered at $d$, where $v(e) = \frac{1}{2}(2n - s + \frac{1}{p-1})$.*

*Proof.* By Lemma 2.26, the ramification invariant $\sigma$ of any étale tail is an integer. Clearly there are no primitive tails, as there are no branch points with prime-to-$p$ branching index. By Lemma 3.8, any new tail has $\sigma \geq 2$. By the vanishing cycles formula (3.1.2), there is exactly one new tail $\overline{X}_b$ and its invariant $\sigma_b$ is equal to 2.

*To (i):* Let $e$ be an element of $\overline{K_0}$ with $v(e) = \frac{1}{2}(n + \frac{1}{p-1})$, and let $K$ be a subfield of $\overline{K_0}$ containing $K_0(\zeta_{p^n})$ and $e$. Let $R$ be the valuation ring of $K$. Consider the smooth model $X'_R$ of $\mathbb{P}^1_K$ corresponding to the coordinate $t$, where $x = d + et$. The formal disk $D$ corresponding to the completion of $D_k = X'_k \backslash \{t = \infty\}$ in $X'_R$ is the closed disk of radius 1 centered at $t = 0$, or equivalently, the disk of radius $|e|$ centered at $x = d$ (§2.4). Its ring of formal functions is $R\{t\}$.

We know that $f$ is given by an equation of the form $y^{p^n} = g(x) := cx^a(x-1)^b$, and that any value of $c$ yields $f$ over $\overline{K_0}$. Taking $K$ sufficiently large, we may assume $c = \frac{(a+b)^{a+b}}{a^a(-b)^b}$. In order to calculate the normalization of $X'_R$ in $K(Y)$, we calculate the normalization $E$ of $D$ in the fraction field of $R\{t\}[y]/(y^{p^n} - g(x)) = R\{t\}[y]/(y^{p^n} - g(d + et))$. Now,

$$g(d + et) = g(d) + \frac{g'(d)}{1!}(et) + \frac{g''(d)}{2!}(et)^2 + \cdots .$$

A quick calculation shows that $g(d) = 1$, $g'(d) = 0$, and $g''(d) = \frac{(a+b)^3}{ab}$ is a unit in $R$. Thus $v(\frac{g''(d)}{2!}e^2) = n + \frac{1}{p-1}$, and the coefficients of the higher powers of $t$ have higher valuations. Since $p \neq 2$, we are in the situation of Corollary 2.34, and the special fiber $\overline{E}$ of $E$ is a disjoint union of $p^{n-1}$ étale covers of $\overline{D} := D \otimes_R k \cong \mathbb{A}^1_k$. By Remark 2.33 (iii), each of these covers extends to an Artin-Schreier cover of conductor 2 over $\mathbb{P}^1_k$. By Corollary 2.13, these have genus $\frac{p-1}{2} > 0$, and thus the component $\overline{X}_b$ corresponding to $D$ is included in the stable model. By Proposition 2.22, it is a tail. Since there is only one tail of $X$, and it has ramification invariant 2, it must correspond to $\overline{X}_b$.

*To (ii):* We repeat the proof of (i), except in this case we choose $e$ such that $v(e) = \frac{1}{2}(2n - s + \frac{1}{p-1})$. Then, letting $g(x) = x^a(x-1)^b$, we have that $g(d) = 1$, $g'(d) = 0$, and $g''(d) = \frac{(a+b)^3}{ab}$ has valuation $s - n$ in $R$. Thus $v(\frac{g''(d)}{2!}e^2) = n + \frac{1}{p-1}$, and the coefficients of the higher powers of $t$ have higher valuations. We conclude as in (i). $\square$

**Corollary 4.11.** *(i) If $f$ is totally ramified above all three branch points, then $\overline{X}$ has no inseparable tails.*

*(ii) If $f$ is totally ramified above only $0$ and $\infty$, then if $\overline{X}$ has an inseparable tail, it contains the specialization of $x = 1$.*

*Proof. To (i):* Suppose there is an inseparable tail $\overline{X}_c \subset \overline{X}$. Suppose $\overline{X}_c$ is a $p^j$-component ($j < n$, by Proposition 2.24). Let $\sigma_c$ be its ramification invariant (page 60). Then, by Proposition 2.19, $\overline{X}_c$ does not contain the specialization of a branch point of $f$. By Lemma 3.8, this means that $\sigma_c > 1$. Then let $Q < G$ be the unique subgroup of order $p^j$. The stable model of the cover $Y/Q \to X$ is a contraction of $Y^{st}/Q$. But $\overline{X}_c$ is a new tail of $Y^{st}/Q$. Since $\sigma_c > 1$, the components of $\overline{Y}/Q$ above $\overline{X}_c$ have positive genus, and thus cannot be contracted in the stable model of $Y/Q \to X$. So $\overline{X}_c$ is a new tail of the stable model of $Y/Q \to X$. But it does not contain the specialization of $d$. This contradicts Lemma 4.10 (substituting $n-j$ for $n$ in the statement), showing that $\overline{X}_c$ cannot exist.

*To (ii):* Assume $f$ is ramified above $x = 1$ of index $p^s$, $s < n$. Again let $\overline{X}_c$ be an inseparable tail of $\overline{X}$ that is a $p^j$-component, and $\sigma_c$ its ramification invariant. Assume that $x = 1$ does not specialize to $\overline{X}_c$. If $j \geq s$, then the generalized vanishing cycles formula (3.1.7) with $r = j$ yields $0 \geq \sigma_c - 1$. But $\sigma_c \geq 2$ by Lemma 3.8, giving a contradiction. Now suppose $j < s$. Letting $Q < G$ be the unique subgroup of order $p^j$, we see that $Y/Q$ is a three-point cover. So we obtain the same contradiction as in (i). □

**Lemma 4.12.** *The stable reduction $\overline{X}$ cannot have a $p^i$-component intersecting a $p^{i+j}$-component, for $j \geq 2$.*

*Proof.* Let $\overline{X}_c$ be such a $p^i$-component, and let $\sigma_c = \sigma_{j,c}$ be its ramification invariant. Then since every term on the right-hand side of the generalized vanishing cycles formula (3.1.7) for $r = i$ is nonnegative, we have that $1 \geq \sigma_c - 1$. But by [Pri06, Lemma 19], $\sigma_c \geq p^{j-1}\sigma_{1,c}$. Since $j \geq 2$ and $p > 2$, then $\sigma_c > 2$. This is a contradiction. □

We now give the structure of the stable reduction when $f$ has three totally ramified points.

**Proposition 4.13.** *Suppose that $f$ is totally ramified above all three branch points. Then $\overline{X}$ is a chain, with one $p^{n-i}$-component $\overline{X}_i$ for each $i$, $0 \leq i \leq n$ ($\overline{X}_0$ is the original component). For each $i > 0$, the component $\overline{X}_{n-i}$ corresponds to the closed disk of radius $|e_i|$ centered at $d = \frac{a}{a+b}$, where $v(e_i) = \frac{1}{2}(i + \frac{1}{p-1})$.*

*Proof.* We know from Lemma 4.10 and Corollary 4.11 that $\overline{X}$ has only one tail, so it must be a chain. The original component contains the specializations of the branch points, so it must be a $p^n$-component. By, Lemma 4.12, there must be a $p^{n-i}$-component for each $i$, $0, \leq i \leq n$. Also, there cannot be two components $\overline{W} \prec \overline{W}'$ with the same inertia, as the components lying above $\overline{W}'$ would be purely inseparable over $\overline{W}'$, with only two marked points. This violates the three-point condition.

It remains to show that the disks are as claimed. For $i = n$, this follows from Lemma 4.10. For $i < n$, consider the cover $Y/Q_i \to X$, where $Q_i$ is the unique subgroup of $G$ of order $p^{n-i}$. The stable model of this cover is a contraction of $Y^{st}/Q_i \to X^{st}$. By Lemma 4.10 (using $i$ in place of $n$), the stable reduction of $Y/Q_i \to X$ has a new tail corresponding to a closed disk centered at $d$ with radius $p^{-\frac{1}{2}(i+\frac{1}{p-1})}$. Thus $\overline{X}$ also contains such a component. This is true for every $i$, proving the proposition. □

Things are more complicated when $f$ has only two totally ramified points:

**Proposition 4.14.** *Suppose that $f$ is totally ramified above $0$ and $\infty$, and ramified of index $p^s$ above $1$, for $0 < s < n$. Then $\overline{X}$ has stable reduction as in Figure 4.1. Furthermore, if $\rho \in \overline{K_0}$ such that $|\rho|$ is the radius of a closed disk corresponding to a component of $\overline{X}$ other than the original component, then $v(\rho) \in \frac{1}{2(p-1)}\mathbb{Z}$.*

*Proof.* We know by Lemma 4.10 that $\overline{X}$ has exactly one étale tail. By Corollary 4.11, $\overline{X}$ can have at most one inseparable tail, which contains the specialization of $x = 1$. We claim that this tail does, in fact, exist. Suppose it did not. Then $\overline{X}$ would be a chain. In particular, there would be a component $\overline{W}$ in this chain to which $x = 1$ specializes. In

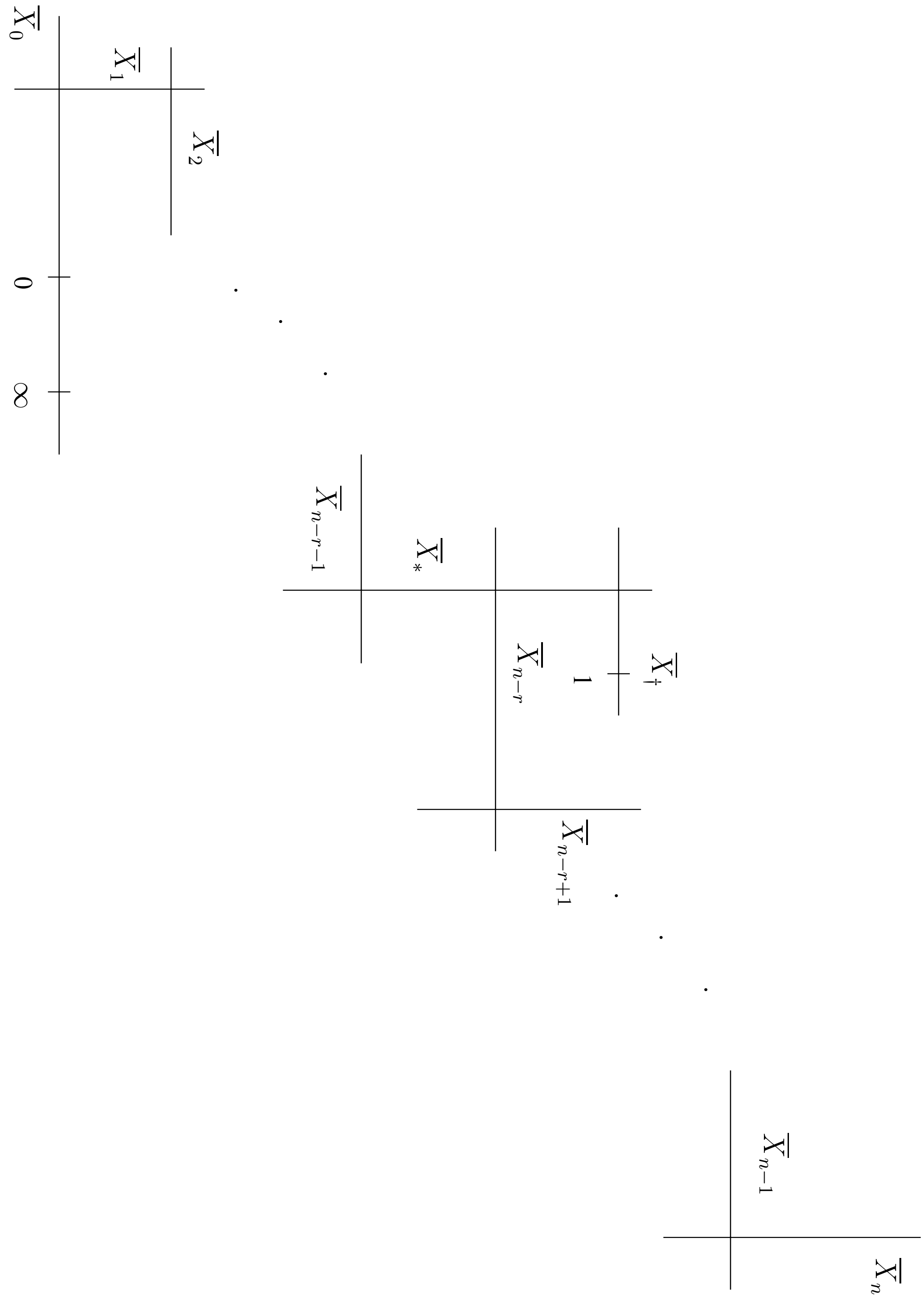


Figure 4.1: The stable reduction with two full branch points

particular, the deformation data above $\overline{W}$ are mulitplicative and identical. Let $\alpha$ (resp. $\beta$) be the intersection of $\overline{W}$ with the immediately preceding (resp. following) component of $\overline{X}$. Let $\sigma_\alpha$ and $\sigma_\beta$ be the invariants of one of the deformation data at $\alpha$ and $\beta$ respectively. By Lemma 2.47, $\sigma_\beta = \sigma_\beta^{\text{eff}} = 2$. Also, if $\gamma$ is the point to which $x = 1$ specializes, then $\sigma_\gamma = 0$. Then the local vanishing cycles formula (2.7.2) yields $\frac{\sigma_\alpha}{p} + A = -1$, where $A$ is some positive number. Thus $\sigma_\alpha$ is negative, which contradicts the fact that the deformation data is multiplicative. This proves the claim.

Recall that $v(1-d) = v(\frac{b}{a+b}) = v(b) = n - s$. Since $\overline{X}$ has a tail containing the specialization of 1 (call this $\overline{X}_\dagger$) and one containing the specialization of $d$, there must be a component of $\overline{X}$ "separating" 1 and $d$, i.e., corresponding to the disk centered at 1 (equivalently, $d$) of radius $|1-d|$. Call this component $\overline{X}_*$. Then $\overline{X}$ looks like a chain from the original component $\overline{X}_0$ to $\overline{X}_*$, followed by two chains, one going out to $\overline{X}_\dagger$ and one going out to the new tail, which we will call $\overline{X}_n$ in this proof.

Let us first examine the part of $\overline{X}$ between $\overline{X}_0$ and $\overline{X}_*$. Let $Q < G$ be the unique subgroup of order $p^s$, and consider the cover $f' : (Y^{st})' := Y^{st}/Q \to X^{st}$. For any singular point $\overline{x}$ of $\overline{X}$ on this chain lying on an intersection of components $\overline{W} \prec \overline{W}'$, we will take $(\sigma_{\overline{x}}^{\text{eff}})'$ to mean the effective invariant for the deformation data above $\overline{W}$ at $\overline{x}$ for the cover $f'$. Now, $f'$ is a cover branched at two points. By (3.1.1), any tail $\overline{X}_b$ of the stable reduction of this graph must have $\sigma_b = 1$. So we always have $(\sigma_{\overline{x}}^{\text{eff}})' = 1$, by Lemma 2.47. Above $\overline{X}_0$, the effective different $(\delta^{\text{eff}})'$ for $f'$ is $n - s + \frac{1}{p-1}$. So above $\overline{X}_*$, it is $n - s + \frac{1}{p-1} - (n-s) = \frac{1}{p-1} > 0$, by Lemma 2.46 applied to all of the singular points between $\overline{X}_0$ and $\overline{X}_*$. This means that $\overline{X}_*$ is an inseparable component for $f'$, which means that it is at least a $p^{s+1}$-component for $f$. On the other hand, if we quotient out further by a group of order $p$, a similar calculation will show that $\overline{X}_*$ becomes an étale component. So $\overline{X}_*$ is a $p^{s+1}$-component.

By Lemma 4.12, there must be a $p^i$-component between $\overline{X}_0$ and $\overline{X}_*$ for each $i$, $s+1 \leq i \leq n$. If we let $Q_i$ be the subgroup of $G$ of order $p^i$, then if we take $f_i' : (Y^{st})_i' := Y^{st}/Q_i \to X^{st}$, the effective different above $\overline{X}_0$ is $n - i + \frac{1}{p-1}$. Since all of the $\sigma^{\text{eff}}$'s are still equal to 1, Lemma 2.46 shows that above a component corresponding to the closed disk of radius $p^{-(n-i+\frac{1}{p-1})}$ centered at $d$, the effective different will be 0. This means that this component is the innermost $p^i$-component. For $i > s+1$, we cannot have two $p^i$-components intersecting each other, because the three-point condition on $\overline{f}$ will be violated on the outermost one (the components lying above would be purely inseparable over it). For $i = s+1$, we have a $p^i$-component corresponding to the closed disk of radius $n - s - 1 + \frac{1}{p-1}$ around $d$ intersecting $\overline{X}_*$, which corresponds to the closed disk of radius $n - s$ around $d$. We label each $p^i$-component in this chain (excepting $\overline{X}_*$), by $\overline{X}_{n-i}$ in Figure 4.1. We have shown that their radii satisfy the condition in the proposition.

Now, let us examine the part of $\overline{X}$ between $\overline{X}_*$ and $\overline{X}_\dagger$. We have seen that $\overline{X}_*$ is a $p^{s+1}$-component, and $\overline{X}_\dagger$ is a $p^s$-component by Proposition 2.19. So this part of $\overline{X}$ consists only of these two components. Recall that if we quotient out $Y^{st}$ by $Q$ (which is $Q_s$), the effective different above $\overline{X}_*$ is $\frac{1}{p-1}$. Also, recall that the effective invariant $\sigma^{\text{eff}}$ at $\overline{X}_* \cap \overline{X}_\dagger$ above $\overline{X}_*$ is 1 after taking this quotient. So by Lemma 2.46, $\overline{X}_\dagger$ corresponds to the disk of radius $p^{-(n-s+\frac{1}{p-1})}$ centered at 1. This satisfies the condition of the proposition.

Lastly, let us examine the part of $\overline{X}$ between $\overline{X}_*$ and the new tail $\overline{X}_n$. We know there must be a $p^i$-component for each $i$, $0 \leq i \leq s+1$. This component must be unique, as if there are two $p^i$-components that intersect, we will again violate the three-point condition above the outermost one. These components are labeled $\overline{X}_{n-i}$ in Figure 4.1 (with the exception of $\overline{X}_*$). We calculate the radius of the closed disk corresponding to each $\overline{X}_{n-i}$. For $i = s$, the radius is $p^{-(n-s+\frac{1}{p-1})}$ for exactly the same reasons as for $\overline{X}_\dagger$. For $i = 0$, we already know from Lemma 4.10 that the radius is $p^{-\frac{1}{2}(2n-s+\frac{1}{p-1})}$. For $1 \leq i \leq s-1$, we consider the cover $Y/Q_i \to X$, where $Q_i$ is the unique subgroup of $G$ of order $p^{n-i}$. The stable model of this cover is a contraction of $Y^{st}/Q_i \to X^{st}$. Since $Y/Q_i \to X$ is still a three-point cover, we can use Lemma 4.10 (with $n-i$ in place of $n$) to obtain that the stable reduction of $Y/Q \to X$ has a new tail corresponding to a closed disk centered at $d$ with radius $p^{-\frac{1}{2}(2n-s-i+\frac{1}{p-1})}$. This is the component $\overline{X}_{n-i}$. Again, these radii all satisfy the condition of the proposition. $\square$

The stable reduction of $f$ having been described, we now turn to the minimal field of definition of the stable model.

**Lemma 4.15.** *Suppose a component $\overline{W}$ of $\overline{X}$ corresponds to a disk of radius $|e|$ containing a $K_0$-rational point $x_0$. Then any element of $G_{K_0(e)}$ acting on $X^{st}$ fixes $\overline{W}$ pointwise.*

*Proof.* Let $\overline{x}$ be a point of $\overline{W}$ which is a smooth point of $\overline{X}$. Then $\overline{x}$ is not the specialization of $\infty \in X$. Choose some lift $x \in \overline{K_0}$. The set of points that specialize to $\overline{x}$ is an open disc of radius $|e|$. Now, if $\overline{x}$ is the specialization of $x_0$, then clearly it is fixed by $G_{K_0(e)}$. If not, then $\frac{x-x_0}{e}$ has valuation 0 in $\overline{K_0}$. Choose $u \in K_0$ such that $v(u - \frac{x-x_0}{e}) > 0$. Then $v((eu + x_0) - x) > v(e)$, and thus $eu + x_0$ also specializes to $\overline{x}$. But $eu + x_0 \in K_0(e)$, so

$G_{K_0(e)}$ fixes $\overline{x}$. By continuity it fixes all of the singular points of $\overline{X}$ on $\overline{W}$. □

We give the major result of this section:

**Proposition 4.16.** *Assume $G = \mathbb{Z}/p^n$, $n \geq 1$, $p \neq 2$, and $f : Y \to X$ is a three-point $G$-cover defined over $\overline{K}_0$.*

*(i) If $f$ is totally ramified above 0, 1, and $\infty$, then there is a model for $f$ defined over $K_n = K_0(\zeta_{p^n})$ whose stable model is defined over $K_n(\sqrt{\zeta_p - 1})$.*

*(ii) If $f$ is totally ramified above 0 and $\infty$, and of index $p^s$ above 1, then there is a model for $f$ over $K_n$ whose stable model is defined over $K_n(\sqrt{\zeta_p - 1}, \sqrt[p^{n-s}]{c})$, where $c \in \mathbb{Q}$ and $p^{n-s} | v_p(c)$.*

*Proof. To (i):* Take $f$ to be given by the equation $y^{p^n} = cx^a(x-1)^b$, with $a, b, a+b$ prime to $p$ and $c = \frac{(a+b)^{a+b}}{a^a(-b)^b}$. Set $K^{st} = K_n(\sqrt{\zeta_p - 1})$. Note that $v(K^{st}) = \frac{1}{2(p-1)}\mathbb{Z}$.

By §2.5, we must show that $G_{K^{st}}$ acts trivially on $\overline{Y}$. First we show that $G_{K^{st}}$ acts trivially on $\overline{X}$. By Proposition 4.13, every component of $\overline{X}$ corresponds to a closed disk containing the $K_0$-rational point $d$ whose radius is the absolute value of an element whose valuation is in $\frac{1}{2(p-1)}\mathbb{Z}$. We can choose such an element in $K^{st}$. So $G_{K^{st}}$ acts trivially on every component of $\overline{X}$, and thus on $\overline{X}$.

So the action of $G_{K^{st}}$ on $\overline{Y}$ must be vertical. Consider the fiber of $Y \to X$ above $d = \frac{a}{a+b}$. From the equation $y^{p^n} = cx^a(x-1)^b$ of our model, the points of this fiber are all rational over $K^{st}$ (they are $p$th roots of unity). Thus their specializations are fixed by $G_{K^{st}}$. Since $d$ specializes to $\overline{X}_n$ and $G_{K^{st}}$ acts vertically, $G_{K^{st}}$ fixes all of $\overline{Y}$ above $\overline{X}_n$. By inward induction and continuity, $G_{K^{st}}$ fixes $\overline{Y}$.

*To (ii):* Take $c$ and the equation for $f$ as in part (i). Note that $v(c) = -bv(b)$. Since $v(b) = n - s$, it follows that $p^{n-s}|v(c)$. We set $G_{K^{st}} = K_n(\sqrt{\zeta_p - 1}, \sqrt[p^{n-s}]{c})$.

As in the proof of (i), this time using Proposition 4.14 instead of 4.13, we see that $G_{K^{st}}$ must act vertically on $\overline{Y}$. Exactly as in (i), we see that $G_{K^{st}}$ acts trivially above the unique étale tail of $\overline{X}$. Now, consider $\overline{Y}/(Q_s)$, where $Q_s$ is the unique subgroup of order $p^s$ in $G$. This is a cover of $X$ given birationally by the equation $y^{p^{n-s}} = cx^a(x-1)^b$. Since $p^{n-r}$ exactly divides $b$, we set $y' = y/(x-1)^{b/p^{(n-s)}}$. The new equation $(y')^{p^{n-s}} = cx^a$ shows that the points above $x = 1$ are defined over the field $K_0(\sqrt[p^{n-s}]{c})$. Thus they are defined over $K^{st}$. Again, by inward induction and continuity, $G_{K^{st}}$ fixes $\overline{Y}$. □

**Corollary 4.17.** *In either case covered in Proposition 4.16, let $K^{st}$ be the minimal field of definition of the stable reduction, and let $\Gamma = Gal(K^{st}/K_0)$. Then $\Gamma^n = \{Id\}$ (where $\Gamma^n$ means the nth higher ramification group for the upper numbering).*

*Proof.* In case (i) of Proposition 4.16, $K^{st}$ is a tame extension of $K_0(\zeta_{p^n})$. The $n$th higher ramification groups for the upper numbering for $K_0(\zeta_{p^n})/K_0$ vanish by [Ser79, Corollary to IV, Proposition 18]. Since the upper numbering is preserved by quotients, the $n$th higher ramification groups vanish for $K^{st}/K_0$ as well.

For case (ii) of Proposition 4.16, we show that $K_0(\zeta_{p^n}, \sqrt[p^{n-s}]{c})/K_0$ has trivial $n$th higher ramification groups for the upper numbering, where $p^{n-s}|v_p(c)$ and $c \in \mathbb{Q}$. Writing $c' = c/p^{v_p(c)}$ we have that $v(c') = 0$ and $K_0(\zeta_{p^n}, \sqrt[p^{n-s}]{c}) = K_0(\zeta_{p^n}, \sqrt[p^{n-s}]{c'})$. Then the result follows from [Viv04, Theorem 5.8]. □

### 4.2.2 The case of a general cover, $m = 1$

**Proposition 4.18.** *Let $G$ be a finite group with a cyclic $p$-Sylow subgroup $P$ of order $p^n$. Assume $N_G(P) = Z_G(P)$ and $p \neq 2$. If $f : Y \to X$ is a three-point $G$-cover of $\mathbb{P}^1$ defined over $\overline{K_0}$, then the field of moduli $K$ of $f$ relative to $K_0$ is a tame extension of $K_n := K_0(\zeta_{p^n})$. Furthermore, there exists a model $f_K$ of $f$ over $K$ such that if $K^{st}/K_0$ is the minimal extension over which the stable reduction of $f_K$ is defined, then the $n$th higher ramification groups for the upper numbering of the extension $K^{st}/K_0$ vanish.*

*Proof.* By Corollary 2.4, we know there is a prime-to-$p$ normal subgroup $N \leq G$ such that $G/N \cong \mathbb{Z}/p^n$. Let $Y^\dagger = Y/N$. If the quotient cover $f^\dagger : Y^\dagger \to X$ is branched at three points, then Lemmas 2.30 and 2.31, combined with Proposition 4.9 and Corollary 4.17, yield the proposition. If, however, $f^\dagger$ is a cyclic $\mathbb{Z}/p^n$-cover branched at only two points, say $x = 0$ and $x = \infty$, then we must mark the specializations of the points lying over $x = 1$, and require these markings to be separated on the closed fiber. Call this model $(f^\dagger)' : ((Y^\dagger)^{st})' \to (X^{st})'$ (cf. §2.6 before Lemma 2.30). We will show that it is defined over $K_n = K_0(\zeta_{p^n})$. Then Lemmas 2.30 and 2.31 will yield the proposition.

We may assume that $f^\dagger : Y^\dagger \to X$ is given by the equation $y^{p^n} = x$, which is defined over $K_n$ as a $\mathbb{Z}/p^n$-cover. Let $R_n$ be the ring of integers of $K_n$, and take the smooth model $X_{R_n}$ of $X$ corresponding to the coordinate $x$. The normalization of $X_{R_n}$ in $K_n(Y^\dagger)$ has smooth, irreducible special fiber $\overline{Y}^\dagger$ that is purely inseparable of degree $p^n$ over the special fiber $\overline{X}_0$ of $X_{R_n}$. All points above $x = 1$ specialize to the same point on $\overline{Y}^\dagger$. In order to separate these points, we take $n$ successive blowups to introduce a chain of intersecting components $\overline{X}_1, \ldots, \overline{X}_n$, where $\overline{X}_i$ is a $p^{n-i}$-component (we cannot have a $p^a$-component intersect a $p^{a-j}$-component with $j > 1$, by Lemma 4.12. This is the model $\overline{X}^{st}$, and its normalization in $K(Y)$, for $K$ a large enough extension of $K_n$, is $((Y^\dagger)^{st})'$. We must show $((Y^\dagger)^{st})'$ is in fact defined over $K_n$.

Each component $\overline{X}_i$, $1 \leq i \leq n$ corresponds to a disk centered at $x = 1$. Let $x_i = \overline{X}_i \cap \overline{X}_{i+1}$. To calculate the radius of the disk corresponding to $\overline{X}_i$, we take the quotient of $((Y^\dagger)^{st})'$ by $\mathbb{Z}/p^{n-i}$. Let $0 \leq j < i$. By Lemma 2.47, $\sigma_{x_j}^{\text{eff}}$ for the deformation data above $\overline{X}_j$ is 1. Also, the effective different above $\overline{X}_0$ for the quotient cover is $i + \frac{1}{p-1}$, whereas the effective different above $\overline{X}_i$ is zero. Then, applying Lemma 2.46 to each of the $x_j$, $0 \leq j < i$, shows that the radius of the disk corresponding to $\overline{X}_i$ is $p^{-(i+\frac{1}{p-1})}$. The valuation of this is $i + \frac{1}{p-1}$.

As in the proof of Lemma 4.15, each smooth point of $\overline{X}_i$, $0 \leq i \leq n$, is the specialization of some point defined over $K_n$. Thus $G_{K_n}$ acts trivially on $\overline{X} = \bigcup_{i=0}^{n} \overline{X}_i$. Since the points above $x = 1$ in $Y^\dagger$ are clearly defined over $K_n$, $G_{K_n}$ acts trivially above $\overline{X}_n$. By inward induction, it acts trivially on $((Y^\dagger)^{st})'$. So the cover $(f^\dagger)'$ is defined over $K_n$, and we are done. $\square$

## 4.3 The case where $m = 2$

Since $m|(p-1)$, we may (and do) assume throughout this section that $p \neq 2$. We break this section up into the cases where the number $\tau$ of branch points of $f : Y \to X = \mathbb{P}^1$ with prime-to-$p$ branching index is 1, 2, or 3. The cases $\tau = 2$ and $\tau = 3$ are quite easy, whereas the case $\tau = 1$ is much more involved. This stems from the appearance of new tails in the stable reduction of $f$ in the case $\tau = 1$. The ideas in the proof of the $\tau = 1$

case should work as well in the $\tau = 0$ case; although we have a partial proof, it is not yet complete, so we do not include it here. Before examining the separate cases, we mention the following easy lemma:

**Lemma 4.19.** *If $m = 2$, and if $f$ has bad reduction, then there are at most two étale tails. Furthermore, for any étale tail $\overline{X}_b$, $\sigma_b \in \frac{1}{2}\mathbb{Z}$ (see §2.5).*

*Proof.* That $\sigma_b \in \frac{1}{2}\mathbb{Z}$ follows by Lemma 2.26. Then, in the vanishing cycles formula (3.1.2), each term on the right-hand side is at least $\frac{1}{2}$, using Lemma 3.8. So there can be at most two terms, corresponding to at most two étale tails. □

The $\tau = 3$ case follows from this lemma:

**Proposition 4.20.** *Assume $f : Y \to X$ is a three-point $G$-cover defined over $\overline{K_0}$ where $G$ has a cyclic $p$-Sylow subgroup $P$ with $m = |N_G(P)/Z_G(P)| = 2$. Suppose that all three branch points of $f$ have prime-to-$p$ branching index. Then $f$ has potentially good reduction. Additionally, $f$ has a model defined over $K_0$, and thus the field of moduli of $f$ with respect to $K_0$ is $K_0$.*

*Proof.* Suppose $f$ has bad reduction. By the fact that the specializations of the ramification points of $f$ cannot coalesce on $\overline{Y}$, we have that each of the three branch points must specialize to an étale primitive tail (Proposition 2.19). But this contradicts Lemma 4.19.

Let $\overline{f} : \overline{Y} \to \overline{X}$ be the reduction of $f$ over $k$. $f$ is tamely branched. Then by [Ful69, Theorem 4.10], there exists a unique deformation of $f_R$ to a cover defined over $R$, where $R$ is the ring of integers of any finite extension $K/K_0$. It follows that $f_{R_0}$ exists, and $f_{R_0} \otimes_{R_0} R \cong f_R$. Thus $f_{R_0} \otimes_{R_0} K_0$ is the model we seek. □

### 4.3.1 The Case $\tau = 2$

From now on, we assume that $f$ has bad reduction (if $f$ has good reduction, we conclude as in Proposition 4.20). Now we consider the case where two branch points have prime-to-$p$ branching index (without loss of generality, we can take these points to be 0 and $\infty$). Then each of these specializes to a primitive tail, and Lemma 4.19 shows that there are no new tails. The ramification invariant for each of these tails is $\frac{1}{2}$ by Lemma 4.19 and the vanishing cycles formula (3.1.2). Then the strong auxiliary cover $f^{str} : Y^{str} \to X$ is a three-point $\mathbb{Z}/p^{\nu} \rtimes \mathbb{Z}/2$-cover, for some $\nu \leq n$. Since we know the branching of this cover, we can determine the form of its equation — it must be given birationally by the equations

$$z^2 = x, y^{p^{\nu}} = \left(\frac{z-1}{z+1}\right)^r$$

for some $r \in \mathbb{Z}$. So as a mere cover, $f^{str}$ is defined over $K_0$, and as a $G^{str} \cong \mathbb{Z}/p^{\nu} \rtimes \mathbb{Z}/2$-cover, it is defined over $K_{\nu}$. By Proposition 4.7, the stable model of $f^{str}$ is defined over a tame extension of $K_{\nu}$ (see the proof of Proposition 4.4 (iii)). Lemmas 2.30 and 2.31 show that the same is also true for the stable model of $f^{aux}$, the standard auxiliary cover. Using Lemma 2.28, the same is true for the stable model of $f$. We have shown the following:

**Proposition 4.21.** *Assume $f : Y \to X$ is a three-point $G$-cover defined over $\overline{K_0}$ where $G$ has a cyclic $p$-Sylow subgroup $P$ with $m = |N_G(P)/Z_G(P)| = 2$. Suppose that two of the three branch points of $f$ have prime-to-$p$ branching index. Then $f$ can be defined over some $K$ which is a tame extension of $K_n$. Indeed, even the stable model of $f$ can be defined over such a $K$. Thus the field of moduli of $f$ relative to $K_0$ is contained in a tame* *extension of $K_n$.*

### 4.3.2 The Case $\tau = 1$

Now we consider the case where only one point, say 0, has prime-to-$p$ branching index. The goal of this (rather lengthy) section is to prove the following proposition:

**Proposition 4.22.** *Assume $f : Y \to X$ is a three-point $G$-cover defined over $\overline{K_0}$ where $G$ has a cyclic $p$-Sylow subgroup $P$ with $m = |N_G(P)/Z_G(P)| = 2$ and $p \neq 3, 5$. Suppose that exactly one of the three branch points of $f$ has prime-to-p branching index. Then $f$ can be defined over some $K$ such that the $n$th higher ramification groups for the upper numbering for $K/K_0$ vanish. Indeed, even the stable model of $f$ can be defined over such a $K$. Thus the $n$th higher ramification group for the upper numbering for the field of moduli of $f$ relative to $K_0$ vanishes.*

The outline of the proof is this: We will take the auxiliary cover of $f$, show that its modified stable model can be defined over a field $K/K_0$ whose $n$th higher ramification groups vanish for the upper numbering, and then conclude via Lemma 2.28 that the stable model of $f$ can be defined over $K$.

We mention that, because $m = 2$, the stable reduction of $f$ is monotonic (Proposition 3.10).

We first deal with the case where there is a primitive tail containing the specialization of 0, but no new tails. Then the vanishing cycles formula (3.1.2) shows that this tail has $\sigma = 1$. Furthermore, if $P^{aux}$ is a $p$-Sylow subgroup of $G^{aux}$, we claim that $N_G(P^{aux}) = Z_G(P^{aux})$. If this were not the case, then the strong auxiliary cover would have Galois group $G^{str} \cong \mathbb{Z}/p^\nu \rtimes \mathbb{Z}/2$, for some $\nu \leq n$, but only one branch point with prime-to-$p$ branching index. Then quotienting out by $\mathbb{Z}/p^\nu$ would yield a contradiction.

Since we are assuming that the stable reduction of $f$ has no new tails, the auxiliary cover $f^{aux} : Y^{aux} \to X$ is branched at either two or three points. If it is branched at three points, we are in the situation of Proposition 4.18, and we immediately conclude that the stable model of $f^{aux}$ (which is the modified stable model) is defined over some $K$ such that the $n$th higher ramification groups of the extension $K/K_0$ vanish. If it is branched at two points, it is a cyclic cover, and we conclude that the modified stable model of $f^{aux}$ is defined over $K_n = K_0(\zeta_{p^n})$ as in the proof of Proposition 4.18. From [Ser79, IV, Corollary to Proposition 18], the $n$th higher ramification groups of $K_n/K_0$ vanish. By Lemma 2.28, Proposition 4.22 is true in this case.

We now come to the main case, where there is a new tail $\overline{X}_b$ and a primitive tail $\overline{X}_{b'}$. We will assume for the remainder of this section that $p \neq 3, 5$ (although it is likely that the main results should hold in the case $p = 3, 5$, see Remark 1.5 (iii)). By the vanishing cycles formula (3.1.2), the primitive tail has $\sigma = 1/2$ and the new tail has $\sigma = 3/2$. It is then clear that the auxiliary cover has four branch points, at 0, 1, $\infty$, and $a$ (keep in mind that we have some freedom in how we choose $a$). Also, the modified stable model of the auxiliary cover is, in fact, the stable model. The strong auxiliary cover then has Galois group $G^{str} \cong \mathbb{Z}/p^\nu \rtimes \mathbb{Z}/2$ for some $\nu \leq n$. Without loss of generality, we can assume that 0 and $a$ are branched of index 2, and 1 and $\infty$ are branched of $p$-power index. The cover $f^{str}$ can be given by the description $(g_0, g_1, g_\infty, g_a)$, where $g_0, g_1$, and $g_\infty$ generate $G^{str}$

and $g_a = (g_0 g_1 g_\infty)^{-1}$; see §2.2. Since $g_0$, $g_1$, and $g_\infty$ must generate $G^{str}$, we see that at least one of $\{1, \infty\}$ must be branched of index $p^\nu$. In particular, by Proposition 2.19, there is a $p^\nu$-component of $\overline{X}$. Because $\overline{f}^{str}$ is monotonic, the original component $\overline{X}_0$ is a $p^\nu$-component.

After a possible application of the transformation $x \to \frac{x}{x-1}$ of $\mathbb{P}^1$, which interchanges 1 and $\infty$ while fixing 0, we can further assume that $a$ does not collide with $\infty$ on the smooth model of $X$ corresponding to the coordinate $x$ (i.e., $|a| \leq 1$). We will start by working over a field $K/K_0$ over which the stable model of $f^{str}$ is defined. Let $R$ be the ring of integers of $K$, and let $\pi$ be a uniformizer of $R$. As always, $v$ is a discrete valuation on $K$ such that $v(p) = 1$.

To prove Proposition 4.22, we will show that the stable model of $f$ is defined over a field $K^{st}$ such that the $n$th higher ramification groups of $K^{st}/K_0$ vanish. This proof will be somewhat involved, so we outline the steps here.

Step 0. We prove some preliminary lemmas.

Step 1. We show that the new branch point $a$ can be chosen to be a $K_0$-rational point. This is the main part of the proof.

Step 2. We deduce that $f^{str}$ can be defined as a mere cover over $K_0(\sqrt{1-a}) = K_0$, and as a $G^{str}$-cover over $K_\nu$.

Step 3. We then show that, if $a \not\equiv 0 \pmod{\pi}$, there are no new inseparable tails. If $a \equiv 0 \pmod{\pi}$, there is exactly one new inseparable tail, and we describe its location.

Step 4. We determine an extension $(K^{str})^{st}$ of $K_\nu$ over which the stable model $(f^{str})^{st}$

of $f^{str}$ can be defined.

Step 5. We show that $(f^{aux})^{st}$ is defined over a tame extension $(K^{aux})^{st}$ of $(K^{str})^{st}$.

Step 6. It follows that $f^{st}$ is defined over a tame extension $K^{st}$ of $(K^{aux})^{st}$.

Step 7. We show that the $\nu$th higher ramification groups for the upper numbering for $K^{st}$ vanish. Since $\nu \leq n$, the $n$th higher ramification groups vanish as well.

**Step 0.** The following is a general structural lemma about $\overline{f}^{str}$:

**Lemma 4.23.** *Every étale tail $\overline{X}_b$ of $\overline{X}$ intersects a $p$-component.*

*Proof.* By Proposition 2.24, $\overline{X}_b$ intersects an inseparable component. If $\overline{X}_b$ intersects a $p^\alpha$-component, with $\alpha > 1$, then Lemma 3.8 shows that $\sigma_b \geq p/2$. Since $p^{\alpha-1}/2 > 2$, and all terms on the right hand side of (3.1.2) are positive, this contradicts the vanishing cycles formula (3.1.2). ◻

**Lemma 4.24.** *The point $x = \infty$ is branched of index $p^\nu$ and specializes to the original component $\overline{X}_0$. If $a \not\equiv 1 \pmod{\pi}$, then $f^{str}$ is branched at $x = 1$ of index $p^\nu$, and $x = 1$ also specializes to the original component.*

*Proof.* Assume for a contradiction that $\infty$ is not branched of index $p^\nu$. Then 1 is branched of index $p^\nu$. Obviously, there are no $p^{\nu+j}$-components for $j > 0$. Since $\overline{f}^{str}$ is monotonic, 1 then specializes to $\overline{X}_0$ by Corollary 3.9. Thus, the deformation data above $\overline{X}_0$ are multiplicative and identical, by Proposition 2.41. By assumption and Proposition 2.19, $\infty$ does not specialize to the original component. Then consider the unique point $\overline{x} \in \overline{X}_0$ such that the specialization $\overline{\infty}$ of $\infty$ satisfies $\overline{x} \prec \overline{\infty}$. Since $|a| \leq 1$, there is no étale tail lying outward from $\overline{x}$, and Lemma 2.47 shows that $\sigma_{\overline{x}}^{\mathrm{eff}} = 0$ for the differential data above $\overline{X}_0$. But this means that $\sigma_{\overline{x}} = 0$ for each deformation datum above $\overline{X}_0$. This contradicts Proposition 2.40. We have thus shown that $\infty$ is branched of index $p^\nu$. By Corollary 3.9, $\infty$ specializes to $\overline{X}_0$.

Now suppose $a \not\equiv 1 \pmod{\pi}$. Assume for a contradiction that 1 does not specialize to the original component. Consider the unique point $\overline{x} \in \overline{X}_0$ such that $x \prec \overline{1}$, the specialization of 1. As in the previous paragraph, $\sigma_{\overline{x}} = 0$ for each deformation datum above $\overline{X}_0$, and we get a contradiction. □

**Lemma 4.25.** *Let $c = \alpha + \frac{\beta}{\sqrt{1-a}}$, $\alpha, \beta, a \in K$. Then if $v(c) > 0$ and $v(\alpha) = 0$, there exists an element $a_0 \in K_0(\alpha, \beta)$ such that $v(a - a_0) = v(c) + 2v(\beta)$.*

*Proof.* Choose $a_0 = 1 - \left(\frac{\beta}{\alpha}\right)^2$. Solving for $a$, we find that $a = 1 - \left(\frac{\beta}{c-\alpha}\right)^2$. Then

$$a - a_0 = \beta^2 \left( \frac{2c\alpha - c^2}{\alpha^2(\alpha - c)^2} \right).$$

Clearly, $v(a - a_0) = 2v(\beta) + v(c)$. □

**Step 1.** We must show that the unique new tail $\overline{X}_b$ of $\overline{X}$ contains the specialization of a $K_0$-rational point. From the construction of the auxiliary cover (§2.6), we know that we can then choose this point as our $a$. We will show the existence of the $K_0$-rational point specializing to $\overline{X}_b$ by calculating the radius of the closed disk corresponding to $\overline{X}_b$, and then showing that only values of $a$ that are within this radius of a $K_0$-rational point $a_0$ can arise as the added branch point of the auxiliary cover of a three-point cover. Then, in the construction of the auxiliary cover, we can replace $a$ with $a_0$.

Let us write down the equations of the cover $f^{str} : Y^{str} \to X^{str}$. Let $Z^{str} = Y^{str}/(\mathbb{Z}/p^\nu)$, and write $\overline{Z}$ for $\overline{Y}/(\mathbb{Z}/p^\nu)$. Then $Z^{str} \to X^{str}$ is a degree 2 cover of $\mathbb{P}^1$'s, branched at 0 and $a$. Therefore, $Z^{str}$ can be given (birationally) over $\overline{K_0}$ by the equation

$$z^2 = \frac{x-a}{x}. \tag{4.3.1}$$

Let $\overline{Z}_b$ be the unique irreducible component of $\overline{Z}$ above $\overline{X}_b$. Since $z = \pm 1$ (resp. $\pm\sqrt{1-a}$) corresponds to $x = \infty$ (resp. $x = 1$), then $Y^{str} \to Z^{str}$ can be given (birationally) over $\overline{K}_0$ by the equation

$$y^{p^\nu} = g(z) := (z+1)^r (z-1)^{p^\nu - r} (z + \sqrt{1-a})^s (z - \sqrt{1-a})^{p^\nu - s} \tag{4.3.2}$$

for some integers $r$ and $s$, which are well-defined modulo $p^\nu$. The branching index of $f^{str}$ at $\infty$ is $p^{\nu - v(r)}$, and at 1 it is $p^{\nu - v(s)}$.

**Lemma 4.26.** *Let $\rho$ (resp. $e$) be an element of $K$ such that $|\rho|$ (resp. $|e|$) is the radius of the disk centered at $x = a$ corresponding to $\overline{X}_b$ (resp. the disk centered at $z = 0$ corresponding to $\overline{Z}_b$).*

*(i) If $a \not\equiv 0, 1 \pmod{\pi}$, then $v(\rho) = \frac{2}{3}(\nu + \frac{1}{p-1})$ and $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1})$.*

*(ii) If $a \equiv 0 \pmod{\pi}$, then $v(\rho) = \frac{2}{3}(\nu + \frac{1}{p-1}) + \frac{v(a)}{3}$ and $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1} - v(a))$.*

*(iii) If $a \equiv 1 \pmod{\pi}$, then $v(\rho) = \frac{2}{3}(\nu + \frac{1}{p-1} + v(1-a))$ and $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1} + v(1-a))$.*

*Proof.* Consider the chain of components $\overline{X}_0 =: \overline{W}_0 \prec \overline{W}_1 \prec \cdots \prec \overline{W}_j \prec \overline{W}_{j+1} := \overline{X}_b$ such that each $\overline{W}_i$ intersects $\overline{W}_{i+1}$ at one point. Let $x_i = \overline{W}_i \cap \overline{W}_{i+1}$. Write $\sigma_i^{\text{eff}}$ for the effective invariant at $x_i$ for the deformation data above $\overline{W}_i$, and write $\epsilon_i$ for the épaisseur of the annulus corresponding to $x_i$.

*To (i):* Suppose $a \not\equiv 0, 1 \pmod{\pi}$. Then $\overline{X}_b$ is the only étale tail lying outward from $x_0$. No branch points with branching index divisible by $p$ lie outward from $x_0$, either. By Lemma 2.47, $\sigma_i^{\text{eff}} = \frac{3}{2}$ for all $0 \leq i \leq j$. Also, the effective different above $\overline{X}_0$ is $\nu + \frac{1}{p-1}$, whereas above $\overline{X}_b$ it is zero. By applying Lemma 2.46 to each $x_i$, $0 \leq i \leq j$, we obtain $v(\rho) = \frac{2}{3}(\nu + \frac{1}{p-1})$. Since $z^2 = \frac{x-a}{x}$, and $x$ is a unit in $R$ for all $x$ specializing to $\overline{X}_b$, we have $v(e) = \frac{1}{2}v(\rho) = \frac{1}{3}(\nu + \frac{1}{p-1})$.

*To (ii):* Suppose $a \equiv 0 \pmod{\pi}$. In order to separate the specializations of $a$ and 0 on the special fiber, there must be a component $\overline{W}$ of $\overline{X}$ corresponding to the closed disk of radius $p^{-v(a)}$ and center 0 (or equivalently, center $a$). Suppose $\overline{W} = \overline{W}_{i_0}$. Then, for $i < i_0$, Lemma 2.47 shows that $\sigma_i^{\text{eff}} = 1$. For $i > i_0$, Lemma 2.47 shows that $\sigma_i^{\text{eff}} = \frac{3}{2}$. By construction, we have $\sum_{i=0}^{i_0-1} \epsilon_i = v(a)$. Applying Lemma 2.46 to each of the points $x_0, \ldots, x_{i_0-1}$, we see that the effective different $\delta_{\text{eff}}$ above $\overline{W}$ is $\nu + \frac{1}{p-1} - v(a)$. Then, applying Lemma 2.46 to each of the points $x_{i_0}, \ldots, x_j$, we see that $\sum_{i=i_0}^{j} \epsilon_i = \frac{2}{3}(\nu + \frac{1}{p-1} - v(a))$. So $v(\rho) = \sum_{i=0}^{j} \epsilon_i = \frac{2}{3}(\nu + \frac{1}{p-1}) + \frac{v(a)}{3}$.

Since $z^2 = \frac{x-a}{x}$, then for any $z$, $v(z) = \frac{1}{2}(v(x-a) - v(x))$. We have shown that $\overline{X}_b$ corresponds to the disk $v(x-a) \geq \frac{2}{3}\left(\nu + \frac{1}{p-1}\right) + \frac{v(a)}{3}$. Since $\overline{X}_b$ is a new tail, 0 does not specialize to this disk. So for any $x$ in this disk, $v(x-a) > v(a)$, so $v(x) = v(a)$. This shows that $v(z) = \frac{1}{2}(v(x-a) - v(a))$. We conclude that the disk corresponding to $\overline{Z}_b$ is the disk centered at 0 with radius $|e|$ such that $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1} - v(a))$.

*To (iii):* Suppose $a \equiv 1 \pmod{\pi}$. In order to separate the specializations of $a$ and 1 on the special fiber, there must be a component $\overline{W}$ of $\overline{X}$ corresponding to the closed disk of radius $p^{-v(1-a)}$ and center 1 (or equivalently, center $a$). Suppose $\overline{W} = \overline{W}_{i_0}$. Then, for $i < i_0$, Lemma 2.47 shows that $\sigma_i^{\text{eff}} = \frac{1}{2}$. For $i > i_0$, Lemma 2.47 shows that $\sigma_i^{\text{eff}} = \frac{3}{2}$. By construction, we have $\sum_{i=0}^{i_0-1} \epsilon_i = v(1-a)$. Applying Lemma 2.46 to each of the points $x_0, \ldots, x_{i_0-1}$, we see that the effective different $\delta_{\text{eff}}$ above $\overline{W}$ is $\nu + \frac{1}{p-1} - \frac{1}{2}v(1-a)$. Then, applying Lemma 2.46 to each of the points $x_{i_0}, \ldots, x_j$, we see that $\sum_{i=i_0}^{j} \epsilon_i = \frac{2}{3}(\nu + \frac{1}{p-1} - \frac{1}{2}v(1-a))$. So $v(\rho) = \sum_{i=0}^{j} \epsilon_i = \frac{2}{3}(\nu + \frac{1}{p-1} + v(1-a))$. As in the case where $a \not\equiv 0, 1 \pmod{\pi}$, $v(e) = \frac{1}{2}v(\rho)$, so $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1} + v(1-a))$. $\square$

Recall that

$$g(z) = (z+1)^r (z-1)^{p^\nu - r} (z + \sqrt{1-a})^s (z - \sqrt{1-a})^{p^\nu - s}.$$

Let $t$ be a coordinate on $\overline{Z}_b$, and let $e \in K$ be such that $|e|$ is the radius of the disk $D$ corresponding to $\overline{Z}_b$. Since $x = a$ corresponds to $z = 0$, we have $z = et$. If $\hat{Y}$, $\hat{Z}$ are the formal completions of $(Y^{str})^{st}$ and $(Z^{str})^{st}$ along their special fibers, then the torsor $\hat{Y} \times_{\hat{Z}} D \to D$ is given generically by the equation

$$y^{p^\nu} = g(0) + \frac{g'(0)}{1!}(et) + \frac{g''(0)}{2!}(et)^2 + \cdots.$$

Possibly after a finite extension of $K$ (so that $g(0)$ becomes a $p^\nu$th power), we can write

$$y^{p^\nu} = 1 + \frac{g'(0)}{1!g(0)}(et) + \frac{g''(0)}{2!g(0)}(et)^2 + \cdots.$$

Now, since $\sigma_b = \frac{3}{2}$, we know that this torsor must split into $p^{\nu-1}$ connected components, each of which has étale reduction and conductor 3. Let $c_i = \frac{g^{(i)}(0)}{i!g(0)}e^i$.

We will have to deal separately with the cases $a \not\equiv 0, 1 \pmod{\pi}$, $a \equiv 1 \pmod{\pi}$, and $a \equiv 0 \pmod{\pi}$.

*The case* $a \not\equiv 0, 1 \pmod{\pi}$:

In this case, $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1})$ by Lemma 4.26. Note that $g(0) = \pm(1-a)^{p^\nu/2}$ has valuation 0. Since $g^{(i)}(0)/i!$ is the coefficient of $z^i$ in $g$, it has non-negative valuation, and thus $v(c_i) \geq \frac{i}{3}(\nu+\frac{1}{p-1})$. It follows that for $p|i$ (and $i \neq 0$), $v(c_i) > \nu+\frac{1}{p-1}$. So for our torsor to have the correct kind of reduction, Lemma 2.14 shows that we must have $v(c_3) = \nu+\frac{1}{p-1}$ and $v(c_1), v(c_2) \geq \nu+\frac{1}{p-1}$. In particular, we must have $v(\frac{g'(0)}{g(0)}) \geq \frac{2}{3}(\nu+\frac{1}{p-1})$. A calculation shows that

$$\frac{g'(0)}{g(0)} = 2r - p^\nu + \frac{2s - p^\nu}{\sqrt{1-a}}.$$

We now apply Lemma 4.25 with $c = \frac{g'(0)}{g(0)}$, $\alpha = 2r - p^\nu$, and $\beta = 2s - p^\nu$. So $K_0(\alpha, \beta) = K_0$. Since the branching index of 1 is $p^\nu$, $s$ is a unit modulo $p$ and $v(\beta) = 0$. Since $v(c) \geq \frac{2}{3}(\nu + \frac{1}{p-1})$, there exists $a_0 \in K_0$ such that $v(a - a_0) \geq \frac{2}{3}(\nu + \frac{1}{p-1})$. But by Lemma 4.26, $v(\rho) = \frac{2}{3}(\nu + \frac{1}{p-1})$, where $|\rho|$ is the radius of the disk corresponding to $\overline{X}_b$. So we have found a $K_0$-rational point $a_0$ specializing to $\overline{X}_b$. Note that the assumption $p \neq 5$ was not necessary in this case.

*The case* $a \equiv 0 \pmod{\pi}$:

In this case, $v(e) = \frac{1}{3}(\nu+\frac{1}{p-1} - v(a))$ by Lemma 4.26. As in the proof of Lemma 4.26, write $\overline{W}$ for the irreducible component of $\overline{X}$ which corresponds to the disk of radius $|a|$ around 0.

**Claim 4.27.** We have $v(a) = v(r+s) \leq \nu - 1$. In particular, $v(a) \in \mathbb{Z}$.

*Proof.* The cover $\overline{Y}^{str} \to \overline{Z}^{str}$ splits completely above the specialization $\overline{z}$ of $z = 0$. Let $t$ be a coordinate corresponding to $\overline{Z}_b$, so that $z = et$. Then $\overline{z}$ corresponds to an open disk $|t| < 1$, and [Ray94, Proposition 3.2.3 (2)] shows that this disk splits into $p^\nu$ disjoint copies in $\overline{Y}^{str}$. Recall that $Y^{str} \to Z^{str}$ is given by the equation

$$y^{p^\nu} = g(z) = (z+1)^r (z-1)^{p^\nu - r}(z + \sqrt{1-a})^s (z - \sqrt{1-a})^{p^\nu - s}.$$

This means that $g(et)$ is a $p^\nu$th power in $R[[t]]$. If $\sum \alpha_i t^i$ is a power series in $R[[t]]$ that is a $p^\nu$th power, the coefficient of $t$ must be divisible by $p^\nu$. So the coefficient of $t$ in $g(et)$, which is $g'(0)e$, has valuation at least $\nu$, and thus $v(\frac{g'(0)}{g(0)}) \geq \nu - v(e) = \frac{2}{3}\nu + \frac{1}{3}v(a) - \frac{1}{3(p-1)}$.

On the other hand, $\frac{g'(0)}{g(0)} = 2r - p^\nu + \frac{2s - p^\nu}{\sqrt{1-a}}$, which can be written as

$$2r + p^\nu + (2s + p^\nu)(1 + \frac{a}{2} + O(a^2)) = 2(r+s) + sa + s(O(a^2)) + p^\nu(2 + O(a^2)),$$

where $O(a^2)$ represents terms whose valuation is at least $2v(a)$. If we assume for the moment that $v(a) < \nu - \frac{1}{2(p-1)}$, then we must have $v(\frac{g'(0)}{g(0)}) \geq \frac{2}{3}\nu + \frac{1}{3}v(a) - \frac{1}{3(p-1)} > v(a)$. Since $v(a^2)$ and $v(p^\nu)$ are both greater than $v(a)$, this means $v(2(r+s) + sa) > v(a)$, so $v(r+s) = v(sa) = v(a)$. Since $v(r+s) \in \mathbb{Z}$, $v(r+s) = v(a) \leq \nu - 1$. If instead, we assume that $v(a) \geq \nu - \frac{1}{2(p-1)}$, then $v(\frac{g'(0)}{g(0)}) \geq \frac{2}{3}\nu + \frac{1}{3}v(a) - \frac{1}{3(p-1)} > \nu - 1$. So $v(2(r+s)) = v(r+s) > \nu - 1$.

So it remains to show that we cannot have both $v(a) \geq \nu - \frac{1}{2(p-1)}$ and $v(r+s) \geq \nu$. Suppose this is the case. Then, by multiplying $g(z)$ by $\frac{(z+1)^{r+s}(z-\sqrt{1-a})^{r+s-p^\nu}}{(z-1)^{p^\nu}}$, which is a $p^\nu$th power, we obtain the equation

$$y^{p^\nu} = \left(\frac{z + \sqrt{1-a}}{z+1}\right)^s \left(\frac{z - \sqrt{1-a}}{z-1}\right)^r.$$

Consider the unique component $\overline{V}$ of $\overline{Z}^{str}$ above $\overline{W}$. This component corresponds to the coordinate $z$. The formal completion of $\overline{V}\backslash\{z = \pm 1\}$ is Spec $C$ where

$$C := \text{Spec } R\{(z-1)^{-1}, (z+1)^{-1}\}.$$

We have $\frac{z+\sqrt{1-a}}{z+1} - 1 = 1 + (\sqrt{1-a}-1)(z+1)^{-1}$. Since $v(\sqrt{1-a}-1) = v(a) > \nu - 1 + \frac{1}{p-1}$, this is a $p^{\nu-1}$st power in $C$ (Remark 2.35). Likewise, $\left(\frac{z-\sqrt{1-a}}{z-1}\right)$ is a $p^{\nu-1}$st power in $C$. So $\left(\frac{z+\sqrt{1-a}}{z+1}\right)^s \left(\frac{z-\sqrt{1-a}}{z-1}\right)^r$ is a $p^{\nu-1}$st power in $C$. But this means that there are at least $p^{\nu-1}$ irreducible components in the inverse image of $\overline{V}\backslash\{z = \pm 1\}$ in $\overline{Y}^{str}$, and thus that many irreducible components of $\overline{Y}^{str}$ above $\overline{V}$.

Now, $\overline{W}$ is not a tail, so it is not an étale component by Proposition 2.22. Also, if $\overline{W}$ is a $p$-component, it must intersect a $p^2$-component by Proposition 3.11 and monotonicity. But then the inertia group above this intersection point is of order divisible by $p^2$, so there cannot be $p^{\nu-1}$ irreducible components above $\overline{V}$. If $\overline{W}$ is a $p^j$-component for $j \geq 2$, then again there cannot be $p^{\nu-1}$ irreducible components above $\overline{V}$. This is a contradiction, proving the claim. □

**Lemma 4.28.** *The valuation* $v\left(\frac{g^{(i)}(0)}{i!g(0)}\right) \geq v(a) - v(i)$.

*Proof.* Since $g(0) = \pm(\sqrt{1-a})^{p^\nu}$, it has valuation 0. Now, $\frac{g^{(i)}(0)}{i!}$ is the coefficient of $z^i$ in $g(z)$. Since $v(a) \geq v(a) - v(i)$, it suffices to look modulo $p^{v(a)}$. Then $g(z)$ is congruent to $(z+1)^{r+s}(z-1)^{2p^\nu - r - s}$. The coefficient of $z^i$ is a sum of terms of the form $\pm\binom{r+s}{j}\binom{2p^\nu - r - s}{i-j}$. It is true in general that $v(\binom{\alpha}{\beta}) \geq v(\alpha) - v(\beta)$. This, combined with the fact that $v(i) \geq \min(v(j), v(i-j))$ shows that the coefficient in question has valuation at least $v(r+s) - v(i)$, which is equal to $v(a) - v(i)$, by Claim 4.27. □

Recall that $c_i := \frac{g^{(i)}(0)}{i!}e^i$. Recall also that we are assuming $p > 5$.

**Corollary 4.29.** *For $i > 3$, $v(c_i) > \nu + \frac{1}{p-1}$.*

*Proof.* By Lemma 4.28, $v(c_i) \geq iv(e) + v(a) - v(i)$. By Lemma 4.26 (ii), $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1} - v(a))$. So

$$v(c_i) \geq \frac{i}{3}(\nu + \frac{1}{p-1} - v(a)) + v(a) - v(i) = \nu + \frac{1}{p-1} + \frac{i-3}{3}(\nu + \frac{1}{p-1} - v(a)) - v(i).$$

Therefore, $v(c_i) > \nu + \frac{1}{p-1}$ whenever $\frac{(i-3)}{3}(\nu + \frac{1}{p-1} - v(a)) \geq v(i)$. By Claim 4.27, $\nu - v(a) \geq 1$. Then for $i > 3$ (and $p > 5$), one can see that $\frac{(i-3)}{3}(\nu + \frac{1}{p-1} - v(a)) \geq v(i)$ always holds. This proves the corollary. □

Now, we will show that the tail $\overline{X}_b$ contains the specialization of a $K_0$-rational point. We let $t$ be a coordinate on $\overline{Z}_b$, and let $\overline{Z}_b$ correspond to a disk $D$. Let $\hat{Y}$ and $\hat{Z}$ be the formal completions of $(Y^{str})^{st}$ and $(Z^{str})^{st}$ along their special fibers. As in the case $a \not\equiv 0, 1 \pmod{\pi}$, we have that, after a finite extension of $K$, the torsor $\hat{Y} \times_{\hat{Z}} D \to D$ can be given generically by the equation

$$y^{p^\nu} = 1 + \frac{g'(0)}{1!g(0)}(et) + \frac{g''(0)}{2!g(0)}(et)^2 + \cdots.$$

Since $\sigma_b = \frac{3}{2}$, we know that this torsor must split into $p^{\nu-1}$ connected components, each of which has étale reduction and conductor 3. Let $c_i = \frac{g^{(i)}(0)}{i!g(0)}e^i$. By Corollary 4.34, $v(c_i) > \nu + \frac{1}{p-1}$ for all $i$ such that $p|i$. So for our torsor to have the correct kind of reduction, Lemma 2.14 shows that we must have $v(c_3) = \nu + \frac{1}{p-1}$ and $v(c_1), v(c_2) \geq \nu + \frac{1}{p-1}$. In particular, we must have

$$v(\frac{g'(0)}{g(0)}) = v(2r - p^\nu + \frac{2s - p^\nu}{\sqrt{1-a}}) \geq \nu + \frac{1}{p-1} - v(e) = \frac{2}{3}(\nu + \frac{1}{p-1}) + \frac{1}{3}v(a).$$

Applying Lemma 4.25 with $c = \frac{g'(0)}{g(0)}$ and $\beta = 2s - p^{\nu}$, there exists $a_0 \in K_0$ such that $v(a - a_0) \geq \frac{2}{3}(\nu + \frac{1}{p-1}) + \frac{1}{3}v(a)$. By Lemma 4.26, this is exactly the valuation of the radius of the disk corresponding to $\overline{X}_b$. So there exists a $K_0$-rational point specializing to $\overline{X}_b$.

**Remark 4.30.** We can, in fact, take this $K_0$-rational point to be $a = 1 - \frac{s^2}{r^2}$. Then the branch of the square root that we choose satisfies $\sqrt{1-a} = -\frac{s}{r}$.

*The case* $a \equiv 1 \pmod{\pi}$. We claim that 1 is branched of index strictly less than $p^{\nu}$. Indeed, if 1 were branched of index $p^{\nu}$, then it would specialize to the original component by Corollary 3.9. But 1 must specialize to a smooth point. Since $a \equiv 1 \pmod{\pi}$, then $a$ would specialize to this same point. But this contradicts the definition of the stable model. Let $p^{\nu_1}$ be the branching index of 1. Then we know $v(s) = v_1 := \nu - \nu_1$. Recall that $\infty$ specializes to $\overline{X}_0$. Also, recall from Lemma 4.26 that $\overline{Z}_b$ corresponds to a closed disk of radius $|e|$, where $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1} + v(1-a))$.

Write $v_0 = v(1-a)$. Then, since the specializations of 1 and $a$ cannot collide on $\overline{X}$, we must have a component $\overline{W}$ of $\overline{X}$ corresponding to the disk of radius $p^{-v_0}$ centered at 1 (or equivalently, at $a$).

**Claim 4.31.** We have $v_0 \leq 2(\nu - 1 + \frac{1}{p-1})$.

*Proof.* Let $Q_{\nu_1} \leq G^{str}$ be the unique subgroup of order $p^{\nu_1}$. Consider the cover $f' : (Y^{str})^{st}/Q_{\nu_1} \to X$. The effective different above the original component $\overline{X}_0$ for this cover is $\nu - \nu_1 + \frac{1}{p-1}$. For any singular point $\overline{x} \in \overline{X}$ such that $\overline{X}_0 \prec \overline{x} \prec \overline{W}$, let $(\sigma_{\overline{x}}^{\text{eff}})'$ be the effective invariant at $\overline{x}$ for the deformation data above the innermost component passing through $\overline{x}$ for the cover $f'$. Since no branch point of $f'$ with index divisible by $p$, and only one branch point with index 2, specializes outward from any such $\overline{x}$, Lemma 2.47 shows that $(\sigma_{\overline{x}}^{\text{eff}})' - 1$ is a sum of elements of the form $\sigma - 1$, where $\sigma > 0$, $\sigma \in \frac{1}{2}\mathbb{Z}$, and $\sigma \in \mathbb{Z}$ for all but one term in the sum. Therefore, $(\sigma_{\overline{x}}^{\text{eff}})' - 1 \geq -\frac{1}{2}$, so $(\sigma_{\overline{x}}^{\text{eff}})' \geq \frac{1}{2}$.

By Corollary 3.9 and monotonicity, $x = 1$ specializes to a component which intersects a component which is inseparable for $f'$. In particular, since $x = 1$ specializes on or outward from $\overline{W}$, it must be the case that any component of $\overline{X}$ lying inward from $\overline{W}$ is inseparable for the cover $f'$. Then, we can apply Lemma 2.46 to each $\overline{X}_0 \prec \overline{x} \prec \overline{W}$ to show that if $\delta$ is the effective different above $\overline{W}$ for $f'$, then $\nu - \nu_1 + \frac{1}{p-1} - \delta \geq \frac{1}{2}v_0$. Since $\delta > 0$ and $\nu_1 \geq 1$, this yields $v_0 \leq 2(\nu - 1 + \frac{1}{p-1})$. □

**Claim 4.32.** We have $v_0 = 2v_1$.

*Proof.* As in the case $a \equiv 0 \pmod{\pi}$, we must have that $v(\frac{g'(0)}{g(0)}e) \geq \nu$, so $v(\frac{g'(0)}{g(0)}) \geq \frac{2}{3}\nu - \frac{1}{3}(\frac{1}{p-1} + v_0)$. Since $v_0 < 2(\nu - 1 + \frac{1}{p-1})$ (Claim 4.31), we see that $v(\frac{g'(0)}{g(0)}) > \frac{2}{3} - \frac{1}{p-1} > 0$. Recall that

$$\frac{g'(0)}{g(0)} = 2r - p^{\nu} + \frac{2s - p^{\nu}}{\sqrt{1-a}}.$$

Since $v(2r - p^{\nu}) = 0$, it follows that $v(\frac{2s - p^{\nu}}{\sqrt{1-a}}) = 0$. Therefore $v(\sqrt{1-a}) = v(2s - p^{\nu}) = v_1$. So $v_0 = v(1-a) = 2v_1$. □

From now on we will write all quantities in terms of $v_1$, not $v_0$. So $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1}) + \frac{2}{3}v_1$. Recall that $g(z) = (z+1)^r(z-1)^{p^{\nu}-r}(z+\sqrt{1-a})^s(z-\sqrt{1-a})^{p^{\nu}-s}$.

**Lemma 4.33.**

- *If $i \leq p^{\nu}$, then $v\left(\frac{g^{(i)}(0)}{i!g(0)}\right) \geq (1-i)v_1 - v(i)$.*
- *If $i > p^{\nu}$, then $v\left(\frac{g^{(i)}(0)}{i!g(0)}\right) \geq -p^{\nu}v_1$.*

*Proof.* Since $g(0) = \pm(\sqrt{1-a})^{p^\nu}$, we see that $v(g(0)) = -p^\nu v_1$. Now, $\frac{g^{(i)}(0)}{i!}$ is the coefficient of $z^i$ in $g(z)$, which clearly always has nonnegative valuation. So we have the lemma in the case $i > p^\nu$. For $i \leq p^\nu$, let us examine the coefficient of $z^i$ in $g(z)$. The lemma will follow if we show that this coefficient has valuation at least $(p^\nu - i + 1)v_1 - v(i)$. If we expand $g$ without combining terms, each coefficient of a $z^i$-term has at least $p^\nu - i$ factors of $\sqrt{1-a}$, and thus valuation at least $v_1(p^\nu - i)$. Any term that has $p^\nu - i + 1$ factors of $\sqrt{1-a}$ has high enough valuation, so we need only be concerned with those terms that have exactly $p^\nu - i$ factors of $\sqrt{1-a}$. Such a term has coefficient

$$\pm(\sqrt{1-a})^{p^\nu - i}\binom{s}{j}\binom{p^\nu - s}{i-j}$$

for some $j \leq i$. But it is true in general that $v(\binom{\alpha}{\beta}) \geq v(\alpha) - v(\beta)$. This, combined with the fact that $v(i) \geq \min(v(j), v(i-j))$, shows that the coefficient in question has valuation at least $(p^\nu - i)v_1 + v_1 - v(i)$. Thus we are done. □

As always, recall that $c_i := \frac{g^{(i)}(0)}{i!}e^i$, and that we assume $p > 5$.

**Corollary 4.34.** *For $i > 3$, $v(\frac{g^{(i)}(0)}{i!g(0)}e^i) > \nu + \frac{1}{p-1}$.*

*Proof.* For simplicity, write $c_i$ for $\frac{g^{(i)}(0)}{i!g(0)}e^i$. If $i \leq p^n$, then Lemma 4.33 shows that $v(c_i) \geq (1-i)v_1 - v(i) + iv(e) = v_1 - v(i) + i(v(e) - v_1)$. Since $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1}) + \frac{2}{3}v_1$, we have that $v(c_i) \geq v_1 - v(i) + \frac{i}{3}(\nu + \frac{1}{p-1} - v_1)$, or rearranging, that

$$v(c_i) \geq \nu + \frac{1}{p-1} + \frac{i-3}{3}(\nu + \frac{1}{p-1} - v_1) - v(i). \tag{4.3.3}$$

Therefore, $v(c_i) > \nu + \frac{1}{p-1}$ whenever $\frac{(i-3)(\nu - v_1)}{3} \geq v(i)$. Recall that $\nu - v_1$ is a positive integer. For $i > 3$ (and $p > 3$), one can see that $\frac{(i-3)(\nu - v_1)}{3} \geq v(i)$ holds. So we have shown the corollary for $i \leq p^\nu$.

If $i > p^\nu$, then again by Corollary 4.34, $v(c_i) \geq -p^\nu v_1 + iv(e)$. Since $i \geq p^\nu + 1$, we can write

$$v(c_i) \geq p^\nu(v(e) - v_1) + v(e) = \frac{p^\nu}{3}(\nu + \frac{1}{p-1} - v_1) + \frac{1}{3}(\nu + \frac{1}{p-1} + 2v_1). \tag{4.3.4}$$

The right-hand side can be rewritten as $\nu + \frac{1}{p-1} + \frac{p^\nu - 2}{3}(\nu + \frac{1}{p-1} - v_1)$. This is greater than $\nu + \frac{1}{p-1}$, because $\nu > v_1$. This proves the corollary for $i > p^\nu$. $\square$

Now, we will show that the tail $\overline{X}_b$ contains the specialization of a $K_0$-rational point. We let $t$ be a coordinate on $\overline{Z}_b$, and let $\overline{Z}_b$ correspond to a disk $D$. Let $\hat{Y}$ and $\hat{Z}$ be the formal completions of $(Y^{str})^{st}$ and $(Z^{str})^{st}$ along their special fibers. As in the case $a \not\equiv 0, 1 \pmod{\pi}$, we have that, after a finite extension of $K$, the torsor $\hat{Y} \times_{\hat{Z}} D \to D$ can be given generically by the equation

$$y^{p^\nu} = 1 + \frac{g'(0)}{1!g(0)}(et) + \frac{g''(0)}{2!g(0)}(et)^2 + \cdots .$$

Since $\sigma_b = \frac{3}{2}$, we know that this torsor must split into $p^{\nu-1}$ connected components, each of which has étale reduction and conductor 3. Let $c_i = \frac{g^{(i)}(0)}{i!g(0)}e^i$. By Corollary 4.34, $v(c_i) > \nu + \frac{1}{p-1}$ for all $i$ such that $p|i$. Then Lemma 2.14 shows that, in order for our torsor to have the correct kind of reduction, we must have $v(c_3) = \nu + \frac{1}{p-1}$ and $v(c_1), v(c_2) \geq \nu + \frac{1}{p-1}$. In particular, we must have

$$v(\frac{g'(0)}{g(0)}) = v(2r - p^\nu + \frac{2s - p^\nu}{\sqrt{1-a}}) \geq \nu + \frac{1}{p-1} - v(e) = \frac{2}{3}(\nu + \frac{1}{p-1} - v_1).$$

Applying Lemma 4.25 with $c = \frac{g'(0)}{g(0)}$ and $\beta = 2s - p^\nu$, there exists $a_0 \in K_0$ such that $v(a - a_0) \geq \frac{2}{3}(\nu + \frac{1}{p-1}) + \frac{4}{3}v_1$. By Lemma 4.26, this is exactly the valuation of the radius of the disk corresponding to $\overline{X}_b$. So there exists a $K_0$-rational point specializing to $\overline{X}_b$.

**Step 2.** Choose $a \in K_0$, using Step 1. We have explicit equations (4.3.1) and (4.3.2) for the cover $f^{str}$, which show immediately that $f^{str}$ is defined as a mere cover over $K_0(\sqrt{1-a})$.

Let $\alpha$ be a generator of $\mathbb{Z}/p^{\nu} \leq G^{str}$, and let $\beta$ be an element of order 2 in $G^{str}$. Since $\alpha^*$ fixes $z$, Equation (4.3.2) shows that $\alpha^*(y) = \zeta_{p^{\nu}}^i y$ for some $i \in \mathbb{Z}$. Also, Equation (4.3.1) shows that $\beta^*(z) = -z$. Writing $g(z)$ for the right-hand side of (4.3.2), we see that $\beta^*(g(z)) = g(z)^{-1}\left((z+1)(z-1)(z+\sqrt{1-a})(z-\sqrt{1-a})\right)^{p^{\nu}}$. Thus $\beta^*(y) = \zeta_{p^{\nu}}^i y^{-1}(z+1)(z-1)(z+\sqrt{1-a})(z-\sqrt{1-a})$. This shows that the action of $G^{str}$ is defined over $K_0(\sqrt{1-a}, \zeta_{p^{\nu}})$. Since $a \in K_0$, $\sqrt{1-a}$ is a degree 2 extension of $K_0$. Since $K_{\nu} = K_0(\zeta_{p^{\nu}})$ contains the unique tame extension of $K_0$ of degree $p-1$, and $2|(p-1)$, we have that $K_0(\sqrt{1-a}, \zeta_{p^{\nu}}) = K_{\nu}$. So $f^{str}$ is defined over $K_{\nu}$ as a $G^{str}$-cover.

**Step 3.** We aim to show that $\overline{X}$ has no new inseparable tails, except in the case $a \equiv 0 \pmod{\pi}$, where it has one.

Set $j$ equal to the least integer such that there exists an inseparable tail $\overline{X}_c$ which is a $p^j$-component. In the language of page 60, the invariant $\sigma_c$ of $\overline{X}_c$ satisfies $\sigma_c \geq 2$, by Lemmas 3.6 and 3.8. Write $(Y^{str})' = Y^{str}/Q_j$, where $Q_j < G^{str}$ is the unique subgroup of order $p^j$, and write $(\overline{f}^{str})' : (\overline{Y}^{str})' \to \overline{X}'$ for the stable reduction of $(\overline{f}^{str})'$. The stable model of $(f^{str})' : (Y^{str})' \to X$ is obtained from $(Y^{str})^{st}/Q_j \to X^{st}$ by contracting all components of the special fiber lying outwards from the $p^j$-components which intersect $p^{j+1}$-components.

For the duration of Step 3, we use the following convention: If $\overline{x}$ is a singular point of $\overline{X}'$, lying on the intersection of two irreducible components $\overline{W}' \prec \overline{W}''$, then $\sigma^{\text{eff}}_{\overline{x}}$ is the effective invariant above $\overline{x}$ for the deformation data above $\overline{W}'$ for the cover $(f^{str})'$. Also, $\epsilon_{\overline{x}}$ is the épaisseur of the annulus corresponding to $x$. For any component $\overline{X}_\ell$ of $\overline{X}'$, we will use the notation $\sigma_\ell$ from page 60. By Lemma 4.23, this is always equal to $\sigma_{1,\ell}$. If $a \equiv 0 \pmod{\pi}$ (resp. $a \equiv 1 \pmod{\pi}$), recall from Step 1 that we have a component $\overline{W}$ of $\overline{X}$ separating $a$ and 0 (resp. $a$ and 1).

**Lemma 4.35.** *(i) $\overline{X}_c$ is the only inseparable tail that is a $p^j$-component. Its invariant $\sigma_c$ is equal to 2.*

*(ii) There are two $p^j$-components $\overline{X}_\beta$ and $\overline{X}_{\beta'}$ of $\overline{X}$, other than $\overline{X}_c$, which intersect $p^{j+1}$-components. We have $\sigma_\beta = \sigma_{\beta'} = \frac{1}{2}$. Also, up to switching indices $\beta$ and $\beta'$, we have $\overline{X}_0 \prec \overline{X}_\beta \prec \overline{X}_b$ and $\overline{X}_0 \prec \overline{X}_{\beta'} \prec \overline{X}_{b'}$.*

*(iii) If $a \equiv 1 \pmod{\pi}$, then $j \leq \nu - v_1 - 1$, where $v_1 = v(\sqrt{1-a})$ as in Step 1. If $a \equiv 0 \pmod{\pi}$, then $j \leq \nu - v(a)$.*

*Proof. To (i) and (ii):* Suppose the inseparable tails that are $p^j$-components are indexed by the set $C$. The vanishing cycles formula (3.1.7) with $r = j$, combined with Lemma 4.23, shows that

$$-2 + |\Pi_{r-j+1}| = \sum_{c' \in C} (\sigma_{c'} - 1) + \sum_{\ell \in B_{j,j+1} \backslash C} \sigma_\ell.$$

By the minimality of $j$, any component in $B_{j,j+1}$ that is not in $C'$ must lie between $\overline{X}_0$ and either $\overline{X}_b$ or $\overline{X}_{b'}$. Thus there are at most two such components $\overline{X}_\ell$, and each must satisfy $\sigma_\ell \geq \frac{1}{2}$. Then the only way that (3.1.7) can be satisfied is if $|\Pi_{j+1}| = 2$, $|C| = \{c\}$, $\sigma_c = 2$, $|B_{j,j+1} \backslash C| = 2$, and $\sigma_\ell = \frac{1}{2}$ for each $\ell \in B_{j,j+1}$.

*To (iii):* From Step 1, we know that $f^{str}$ is ramified above 1 of index $p^{\nu - v_1}$ in the case $a \equiv 1 \pmod{\pi}$. From the proof of (ii), $|\Pi_{j+1}| = 2$, which means that 1 is branched of order at least $p^{j+1}$. It follows that $j \leq \nu - v_1 - 1$.

If $a \equiv 0 \pmod{\pi}$, recall that $\overline{W}$ is the component of $\overline{X}$ separating 0 and $a$. Then $\overline{W} \prec \overline{X}_\beta$, as otherwise we would have $\overline{X}_\beta = \overline{X}_{\beta'}$. Consider the cover $(f^{str})'$. We claim that the order of generic inertia above $\overline{W}$ is at most $p^{\nu - v(a)+1}$. If this is true, then $\overline{X}_\beta$, having less inertia than $\overline{W}$, must be a $p^j$-component for $j \leq \nu - v(a)$.

To prove the claim, it suffices to show that $g(z)$ is a $p^{v(a)-1}$st power in

$$A := R\{(z-1)^{-1}, (z+1)^{-1}\}$$

(cf. the proof of Claim 4.27). We may multiply $g(z)$ by $\frac{(z+1)^{r+s}(z-\sqrt{1-a})^{r+s-p^\nu}}{(z-1)^{p^\nu}}$, which is a $p^{v(r+s)}$th power, thus a $p^{v(a)}$th power, by Claim 4.27. We obtain the equation

$$y^{p^\nu} = \left(\frac{z+\sqrt{1-a}}{z+1}\right)^s \left(\frac{z-\sqrt{1-a}}{z-1}\right)^r.$$

As in the proof of Claim 4.27, we see that $\frac{z+\sqrt{1-a}}{z+1} - 1 = 1 + (\sqrt{1-a}-1)(z+1)^{-1}$. Since $v(\sqrt{1-a}-1) = v(a) > v(a) - 1 + \frac{1}{p-1}$, this is a $p^{v(a)-1}$st power in $A$ (Remark 2.35). Likewise, $\left(\frac{z-\sqrt{1-a}}{z-1}\right)$ is a $p^{v(a)-1}$st power in $A$. So $g(z)$ is a $p^{v(a)-1}$st power in $A$, and we are done. $\square$

By Lemma 4.35, the stable reduction $\overline{X}'$ has étale tails $\overline{X}_\beta, \overline{X}'_\beta$, and $\overline{X}_c$ with $\sigma_\beta = \sigma_{\beta'} = \frac{1}{2}$ and $\sigma_c = 2$. The effective different $\delta^{\text{eff}}$ above $\overline{X}_0$ for $(f^{str})'$ is $\nu - j + \frac{1}{p-1}$.

**Lemma 4.36.** *Let $\overline{Z}_\beta$ be the unique component of $\overline{Z}^{str}$ above $\overline{X}_\beta$. Let $e'$ be such that the radius of the disk corresponding to $\overline{Z}_\beta$ is $|e'|$.*

*(i) There exists $q \in \mathbb{Q}$ such that every point $\overline{x} \in X$ specializing to $\overline{X}_c$ satisfies $v(x-a) = q$.*

*(ii)*

- *If $a \not\equiv 0, 1 \pmod{\pi}$, then $v(e') = \nu - j + \frac{1}{p-1} - q$ and $q < v(\rho') = 2v(e')$. Thus $v(e') > \frac{1}{3}(\nu - j + \frac{1}{p-1})$.*
- *If $a \equiv 0 \pmod{\pi}$, then $v(e') = \nu - j + \frac{1}{p-1} - q$ and $q < v(\rho') = 2v(e') + v(a)$. Thus $v(e') > \frac{1}{3}(\nu - j + \frac{1}{p-1} - v(a))$.*
- *If $a \equiv 1 \pmod{\pi}$, then $v(e') = \nu - j + \frac{1}{p-1} - q + v(a)$ and $q < 2v(e')$. Thus $v(e') > \frac{1}{3}(\nu - j + \frac{1}{p-1} + v(a))$.*

*Proof. To (i):* We know that $x = 0$ does not specialize to $\overline{X}_c$. So if $x_1$ and $x_2$ specialize to $\overline{X}_c$, then $v(x_1 - x_2)$ is greater than both $v(x_1 - a)$ and $v(x_2 - a)$. We conclude that $v(x_1 - a) = v(x_2 - a)$. Clearly $q < v(e')$, as if $v(x - a) \geq e'$, then $x$ specializes to $\overline{X}_\beta$.

*To (ii):* The proof of each case uses the same method, so we prove the case $a \equiv 0 \pmod{\pi}$ and leave the others as exercises. Suppose that $q > v(a)$. Then $v(\rho') = q + v(\rho') - q$. Let $\overline{w} \in \overline{X}$ be the maximal point such that $\overline{w} \prec \overline{X}_c$ and $\overline{w} \prec \overline{X}_\beta$. Consider the singular points $\overline{x}$ of $\overline{X}$ such that $\overline{X}_0 \prec \overline{x} \prec \overline{X}_\beta$. By Lemma 2.47 we have the following facts: If $\overline{w} \prec \overline{x}$, then $\sigma_{\overline{x}}^{\text{eff}} = \frac{1}{2}$. If $\overline{W} \prec \overline{x} \preceq \overline{w}$, then $\sigma_{\overline{x}}^{\text{eff}} = \frac{3}{2}$. If $\overline{x} \prec \overline{W}$, then $\sigma_{\overline{x}}^{\text{eff}} = 1$. Using Lemma 2.46, one shows that $v(\rho') - q = 2(\nu - j + \frac{1}{p-1} + \frac{1}{2}v(a) - \frac{3}{2}q)$. Thus $v(\rho') = 2(\nu - j + \frac{1}{p-1} + \frac{1}{2}v(a) - q)$. Since $v(\rho') = 2v(e') + v(a)$ (cf. proof of Lemma 4.26 (ii)), we have $v(e') = \nu - j + \frac{1}{p-1} - q$. If instead, $q < v(a)$, we do a similar computation, this time writing $v(\rho') = v(a) + v(\rho') - v(a)$. □

**Proposition 4.37.** *There are no new inseparable tails unless $a \equiv 0 \pmod{\pi}$. In this case, any inseparable tail must be a $p^{\nu - v(a)}$-component.*

*Proof.* If $t$ is a coordinate corresponding to $\overline{Z}_\beta$, we can write the equation of $(Y^{str})' \to Z^{str}$ in terms of the coordinate $t$ as

$$y^{p^{\nu-j}} = g(e't)/g(0) = 1 + \frac{g'(0)}{1!g(0)}(e't) + \frac{g''(0)}{2!g(0)}(e't)^2 + \cdots .$$

We first claim that if $a \not\equiv 0 \pmod{\pi}$, then the right-hand side is a $p^{\nu-j}$th power in $R\{t\}$. This means that the cover splits into $p^{\nu-j}$ irreducible components above $\overline{Z}_\beta$, each isomorphic to $\overline{Z}_\beta$, which is a contradiction. To prove the claim, we will show that each coefficient $c_i'$ of $t^i$ in $g(e't)/g(0)$, for $i > 1$, has valuation greater than $\nu - j + \frac{1}{p-1}$. By the binomial theorem, $g(e't)/g(0)$ will be a $p^{\nu-j}$th power. Let us note by Lemma 4.36 that, in all cases, if $c_i$ is defined as in Step 1, then $v(c_i') > v(c_i) - \frac{i}{3}j$.

In the case $a \not\equiv 0, 1 \pmod{\pi}$, it is clear from Lemma 4.36 that $v(c_i) > \nu - j + \frac{1}{p-1}$ for $i \geq 3$. Now, let $e$ be such that $v(e) = \frac{1}{3}(\nu + \frac{1}{p-1})$. We know from Step 1 that $v(c_i) = v(\frac{g^{(i)}(0)}{i!g(0)}e^i) \geq \nu + \frac{1}{p-1}$ for $i = 1, 2$. Then

$$v(c_i') = \nu + \frac{1}{p-1} - i(v(e) - v(e')) > \nu + \frac{1}{p-1} - \frac{i}{3}j > \nu - j + \frac{1}{p-1}$$

for $i = 1, 2$. This finishes the case $a \not\equiv 0, 1 \pmod{\pi}$. Note that the assumption $p \neq 5$ was unnecessary here.

If $a \equiv 1 \pmod{\pi}$, then for $i \geq 3$, we have $v(c_i') > v(c_i) - \frac{i}{3}j$. By Equation (4.3.3), we have

$$v(c_i') \geq \nu - j + \frac{1}{p-1} + \frac{i-3}{3}(\nu - j + \frac{1}{p-1} - v_1) - v(i)$$

if $i \leq p^\nu$. By Lemma 4.35, $v_1 \leq \nu - j - 1$. Since $\frac{i-3}{3} > v(i)$ whenever $i \geq 3$ and $p > 5$, we have $v(c_i') > \nu - j + \frac{1}{p-1}$. For $i = 1, 2$, the proof is identical to the case $a \not\equiv 0, 1 \pmod{\pi}$. For $i > p^n$, Equation (4.3.4) gives that

$$v(c_i') \geq \nu - j + \frac{1}{p-1} + \frac{p^\nu - 2}{3}(\nu - j + \frac{1}{p-1} - v_1).$$

Since $\nu - j > v_1$, we again have that $v(c_i') > \nu - j + \frac{1}{p-1}$.

Lastly, suppose $a \equiv 0 \pmod{\pi}$. By Lemma 4.28, we have $v(c_i) \geq v(a) - v(i) + iv(e)$, so $v(c_i') \geq v(a) - v(i) + iv(e')$. By Lemma 4.36, this is equal to

$$\nu - j + \frac{1}{p-1} + (i-1)(\nu - j + \frac{1}{p-1} - v(a)) - v(i).$$

Since $j < \nu - v(a)$ (Proposition 3.13), this is always greater than $\nu - j + \frac{1}{p-1}$ for $i \geq 3$ and $p > 5$ except when $i = p$ and $j = \nu - v(a)$. For $i = 1, 2$, the proof is again identical to the case $a \not\equiv 0, 1 \pmod{\pi}$. □

**Proposition 4.38.** *In the case $a \equiv 0 \pmod{\pi}$, there is, in fact, a new inseparable tail $\overline{X}_c$ which is a $p^j$-component with $j = \nu - v(a)$. Furthermore, $\overline{X}_c$ corresponds to the disk of radius $p^{-(v(a)+\frac{1}{2(p-1)})}$ around $x = \frac{a}{2}$. The two components of $\overline{Z}^{str}$ lying above $\overline{X}_c$ correspond to the disks of radius $p^{-(\frac{1}{p-1})}$ around $z = \pm\sqrt{1}$.*

*Proof.* Recall that the equations for $(Y^{str})' \to X$ are given by

$$z^2 = \frac{x-a}{x}$$

and

$$y^{p^{\nu-j}} = g(z) = (z+1)^r(z-1)^{p^\nu - r}(z+\sqrt{1-a})^s(z-\sqrt{1-a})^{p^\nu - s}.$$

Since $\nu - j = v(a)$ and $p^{v(a)}|(r+s)$ by Claim 4.27, we may multiply $g(z)$ by an $(r+s)$th power. Multiplying by $(\frac{z+1}{z-1})^{r+s}(z-1)^{-p^\nu}(z-\sqrt{1-a})^{-p^\nu}$, we may assume

$$g(z) = \left(\left(\frac{z+\sqrt{1-a}}{z+1}\right)\left(\frac{z-1}{z-\sqrt{1-a}}\right)\right)^s.$$

By Remark 4.30, $1-\sqrt{1-a} = -\frac{r+s}{r}$. For ease of notation, write $\mu = 1-\sqrt{1-a}$. Then $v(\mu) = v(a)$. Then

$$g(z) = \left((1+\frac{\mu}{z+1})(1+\frac{\mu}{z-1-\mu})\right)^s,$$

or

$$g(z) = 1 + 2s\mu\frac{z}{z^2-1} + O(\mu^2). \tag{4.3.5}$$

This will be a more useful choice of $g(z)$ for our purposes.

Let $\overline{X}_c$ be as in the theorem. Let $(X^{st})_c$ be the minimal modification of $X^{st}$ that contains the component $\overline{X}_c$, and let $t$ be a coordinate corresponding to $\overline{X}_c$. We will show that the normalization of $X_c^{st}$ in $K((Y^{str})')$ has special fiber above $\overline{X}_c$ consisting of $2(p^{\nu-j-1})$ irreducible components, each an Artin-Schreier cover of conductor 2. These each have genus $\frac{p-1}{2} > 0$, by Corollary 2.13. But this means that $\overline{X}_c$ must have been part of the stable model to begin with, as these components cannot be contracted in any stable model.

Let us prove the claim. It is easy to see from the equation $z^2 = \frac{x-a}{x}$ that there are two irreducible components of $\overline{Z}^{str}$ lying above $\overline{X}_c$, corresponding to the disks of radius $|e''|$ around $\pm-1$, where $v(e'') = \frac{1}{2(p-1)}$. We pick a square root and call it $\sqrt{-1}$, and we call the corresponding disk $D$. If $\hat{Y}'$, $\hat{Z}$ are the formal completions of $((Y^{str})')^{st}$ and $(Z^{str})^{st}$ along their special fibers, then after a possible finite extension of $K$, the torsor

$\hat{Y} \times_{\hat{Z}} D \to D$ is given generically by the equation

$$y^{p^{\nu-j}} = 1 + \frac{g'(\sqrt{-1})}{1!g(\sqrt{-1})}(e''t) + \frac{g''(\sqrt{-1})}{2!g(\sqrt{-1})}(e''t)^2 + \cdots .$$

Let $d_i = \frac{g^{(i)}(\sqrt{-1})}{i!g(\sqrt{-1})}(e'')^i$. For all $i > 0$, Equation (4.3.5) shows that $v(\frac{g^{(i)}(\sqrt{-1})}{i!g(\sqrt{-1})}) \geq v(\mu) = v(a) = \nu - j$. So for $i > 2$, $v(d_i) > \nu - j + \frac{1}{p-1}$. For $i = 1$, $d_i = \frac{g'(\sqrt{-1})}{g(\sqrt{-1})} = O(\mu^2)$ by direct calculation, so $v(d_1) \geq 2v(\mu) = 2v(a) > \nu - j + \frac{1}{p-1}$. For $i = 2$, $g''(z) = 2s\mu\frac{2z}{(z^2-1)^3}\left(-4(1+z^2) + 2z(z^2-1)\right)$ by direct calculation. Then $v(g''(\sqrt{-1})) = v(\mu) = \nu - j$. Since $v(e'') = \frac{1}{2(p-1)}$ and $v(g(\sqrt{-1}) = 0$, we have that $v(d_2) = \nu - j + \frac{1}{p-1}$. By Corollary 2.34, the special fiber of $\hat{Y} \times_{\hat{Z}} D$ splits into $p^{\nu-j-1}$ Artin-Schreier covers of conductor 2. This proves the claim, completing the proof of the Proposition. □

**Step 4.** From Step 2, we know that $f^{str}$ is defined over $K^{\nu}$ as a $G^{str}$-cover. Recall from §2.5 that the minimal extension $(K^{str})^{st}/K_{\nu}$ over which the stable model of $f^{str}$ is defined is the extension cut out by the subgroup $\Gamma^{st} \leq G_{K_{\nu}}$ that acts trivially on $\overline{f}^{str}$.

**Lemma 4.39.** *The action of $G_{K_{\nu}}$ on $\overline{X}$ is of prime-to-$p$ order.*

*Proof.* Let $\gamma \in G_{K_{\nu}}$ act on $\overline{X}$ with order $p$. We know that both $\overline{X}_b$ and $\overline{X}_{b'}$ contain the specialization of a $K_0$-rational point (namely, $a$ and 0), which is fixed by $\gamma$. Since $\gamma$ must fix the unique singular point of $\overline{X}$ lying on any tail that it acts on, $\gamma$ must fix two points of each étale tail. Since $\mathbb{P}^1_k$ has no automorphisms of order $p$ fixing two points, $\gamma$ acts trivially on the étale tails. In the case $a \equiv 1 \pmod{\pi}$, there is an inseparable tail containing the specialization of the $K_0$-rational point $x = 1$. So $\gamma$ acts trivially on this tail as well. In the case $a \equiv 0 \pmod{\pi}$, Proposition 4.38 tells us that the inseparable tail contains the specialization of the $K_0$-rational point $x = \frac{a}{2}$. Again, $\gamma$ acts trivially on this tail. By Step 3, these are the only possible inseparable tails, so $\gamma$ acts trivially on all tails.

Since $\gamma$ acts trivially on the original component $\overline{X}_0$ and on all tails, it must fix every singular point of $\overline{X}$ (each such point is determined by the set of tails it precedes and the fact that it is singular). Since each interior component contains at least two of these points, $\gamma$ must fix each of these components pointwise. So $\gamma$ fixes $x$ pointwise, proving the lemma. □

**Lemma 4.40.** *If $\gamma \in G_{K_\nu}$ acts on $\overline{Y}$ with order $p$ and acts trivially above all tails of $\overline{X}$, then $\gamma$ acts trivially on $\overline{Y}^{str}$.*

*Proof.* Note that, by Lemma 4.39, the action of $\gamma$ on $\overline{Y}$ is vertical. We proceed by inward induction. Let $\overline{W}$ be a component of $\overline{X}$ intersecting a tail, and let $\overline{V}$ be a component of $\overline{Y}^{str}$ lying above $\overline{W}$. Since $\gamma$ acts trivially above tails of $\overline{X}$, it must fix all of the intersection points of $\overline{V}$ with a component, say $\overline{V}'$, above a tail of $\overline{X}$. Let $\overline{y}$ be such a point. Write $H = \mathrm{Aut}(\overline{V}/\overline{W})$. If $\gamma$ acts nontrivially on $\overline{V}$, then the inertia group $I_{\overline{y}} \leq H$ has order divisible by $p$. But since $\overline{f}^{str}$ is monotonic, the inertia group of $\overline{V}$ contains the inertia group of $\overline{V}'$, and Proposition 2.17 (ii) shows that $I_y$ must be prime to $p$. So $\gamma$ acts trivially. Inducting inwardly to the original component proves the lemma. □

**Proposition 4.41.** *(i) If $a \not\equiv 0, 1 \pmod{\pi}$, then the stable model of $f^{str}$ can be defined over a tame extension $(K^{str})^{st}$ of $K_\nu$.*

*(ii) If $a \equiv 0$ or $1 \pmod{\pi}$ then the stable model of $f^{str}$ can be defined over a tame*

*extension* $(K^{str})^{st}$ *of* $K_\nu(\sqrt[p]{1+u})$*, where* $u \in K_\nu$ *satisfies* $v(u) = 1$.

*Proof.* Let $\gamma \in G_{K_\nu}$ act on $\overline{Y}^{str}$ with order $p$. We show that $\gamma$ acts trivially above the étale tails of $\overline{X}$. The new tail $\overline{X}_b$ contains the specialization of $x = a$. Then the component $\overline{Z}_b$ lying above $\overline{X}_b$ contains the specialization of $z = 0$, by Equation (4.3.1). Now, in Equation (4.3.1), we have $g(0) = \pm(\sqrt{1-a})^{p^\nu}$. This means that the points of $Y^{str}$ above $z = 0$ (thus $x = a$) are all $K_0$-rational, as they are defined by the equation $y^{p^\nu} = g(0)$. Similarly, $\overline{X}_{b'}$ contains the specialization of $x = 0$, which corresponds to $z = \infty$. In Equation 4.3.2, we can multiply $g(z)$ by a $p^\nu$th power to get an alternate equation

$$y^{p^\nu} = \left(\frac{z+1}{z-1}\right)^r \left(\frac{z+\sqrt{1-a}}{z-\sqrt{1-a}}\right)^s$$

for the cover $Y^{str} \to Z^{str}$. Plugging $\infty$ into the right-hand side of this equation gives $y^{p^\nu} = 1$, which shows that the points of $Y^{str}$ above $z = \infty$ (thus $x = 0$) are all $K_0$-rational. Thus, $\gamma$ acts trivially above the étale tails of $\overline{X}$. Since in the case $a \not\equiv 0, 1 \pmod{\pi}$, the étale tails are the only tails (Proposition 4.37), Lemma 4.40 allows us to conclude (i).

Now, consider the case $a \equiv 0 \pmod{\pi}$. By Proposition 4.38, there is an inseparable tail $\overline{X}_c$ which is a $p^{\nu - v(a)}$-component containing the specialization of $x = \frac{a}{2}$. Then each component of $\overline{Z}^{str}$ above $\overline{X}_c$ contains the specialization of one of $z = \pm\sqrt{-1}$. Consider the cover $(Y^{str})' \to Z^{str}$, where $(Y^{str})' = Y^{str}/Q_{\nu - v(a)}$ and $Q_{\nu - v(a)}$ is the unique subgroup of $G^{str}$ of order $p^{\nu - v(a)}$. Equation (4.3.5) shows that this cover can be given by the equation

$$y^{p^{v(a)}} = 1 + 2s\mu\frac{z}{z^2-1} + O(\mu^2),$$

where $v(\mu) = v(a)$. Plugging in $z = \pm\sqrt{-1}$, we get that $y^{p^{v(a)}} = 1 + \alpha$, with $v(\alpha) = v(a)$. Now, Remark 2.35 shows that $1 + \alpha$ has a $p^{v(a)-1}$st root in $R_\nu$, which is of the form

$1+u$ with $v(u)=1$. Then $G_{K_\nu}(\sqrt[p]{1+u})$ acts trivially above $\overline{X}_c$ for this cover. Since the quotient by $Q_{\nu-v(a)}$ is radicial above $\overline{X}_c$, it follows that $\gamma \in G_{K_\nu}(\sqrt[p]{1+u})$ fixes the fiber of $\overline{Y}^{str}$ above $x = \frac{a}{2} \in \overline{X}_c$ pointwise. By Lemma 4.39, if $\gamma$ acts on $\overline{Y}^{str}$ with order $p$, then $\gamma$ acts trivially above $\overline{X}_c$. Then (ii) follows from Lemma 4.40 when $a \equiv 0 \pmod{\pi}$.

Lastly, consider the case $a \equiv 1 \pmod{\pi}$. There is an inseparable tail $\overline{X}_c$ containing the specialization of $x=1$. By Proposition 2.19, $\overline{X}_c$ is a $p^{\nu-v(s)}$-component. Then each component of $\overline{Z}^{str}$ above $\overline{X}_c$ contains the specialization of one of $z = \pm\sqrt{1-a}$. Consider the cover $(Y^{str})' \to Z^{str}$, where $(Y^{str})' = Y^{str}/Q_{\nu-v(s)}$ and $Q_{\nu-v(s)}$ is the unique subgroup of $G^{str}$ of order $p^{\nu-v(s)}$. After multiplying by $p^{v(s)}$th powers, this cover can be given by the equation

$$y^{p^s} = \left(\frac{z+1}{z-1}\right)^r = \left(\frac{2z}{z-1} - 1\right)^r.$$

Recall that, by Lemma 4.32, we have $v(s) = v(\sqrt{1-a})$. Plugging in $z = \pm\sqrt{1-a}$ and multiplying by $-1$, which is a $p^s$th power in $K_\nu$, we get that $y^{p^{v(s)}} = 1+\alpha$, with $v(\alpha) = v(s)$. We conclude (ii) as in the case $a \equiv 0 \pmod{\pi}$. □

**Step 5.** This is an immediate consequence of Lemmas 2.30 and 2.31.

**Step 6.** This is immediate from Lemma 2.28.

**Step 7.** Using Steps 5 and 6, it suffices to show that the $\nu$th higher ramification groups for the extension $(K^{str})^{st}/K_0$ vanish, where $(K^{str})^{st}$ is defined as in Proposition 4.41. Now, Step 4 shows that $(K^{str})^{st}$ is contained in a tame extension of $K_\nu(\sqrt[p]{1+u})$, for some $u$ such that $v(u)=1$. This is the compositum of $K_\nu$ and $K_1(\sqrt[p]{1+u})$. By [Ser79, IV, Proposition 18], the $\nu$th higher ramification group for the upper numbering vanishes for $K_\nu/K_0$. By Example 2.9, the first higher ramification group for the upper numbering vanishes for $K_1(\sqrt[p]{1+u})/K_0$. By Lemma 2.8 the $i$th higher ramification group for $(K^{str})^{st}/K_0$ for the upper numbering vanishes when $i \geq \max(1,\nu) = \nu$. Since $n \geq \nu$, the $n$th higher ramification group vanishes as well. This concludes Step 7, and thus the proof of Proposition 4.22. □

The proof of Theorem 1.4 follows immediately from combining Propositions 4.1, 4.8, 4.9, 4.20, 4.21, and 4.22. □

# Chapter 5

# Further Investigation

I would like to examine the following questions in the near future:

**Question 5.1.** Does Theorem 1.4 hold in the $m = 2$ case even when we allow $p = 3$, $p = 5$, or no prime-to-$p$ branch points?

I expect techniques similar to those used in §4.3 to work in this case. When there are no prime-to-$p$ branch points, the issue is that there can be up to two new tails. In the case of two new tails, we would ideally like to show that each of them contains the specialization of a $K_0$-rational point, and that there are no new inseparable tails. Then we could arrive at the same conclusion as in §4.3.

Now, in the case of two new tails, the strong auxiliary cover $f^{str} : Y^{str} \to X$ contains two branch points $a$ and $b$ of branching index 2, and three branch points 0, 1, and $\infty$ of $p$-power branching index. If none of the specializations of any of the five branch points coalesce on the smooth model of $X$ corresponding to the original component of the stable reduction, and if in addition there are no inseparable tails, then we can in fact show that each new tail contains the specialization of a $K_0$-rational point, using essentially the same techniques as in §4.3. However, there are many more cases to consider.

If we allow $p = 3$ (even if we assume that there is a branch point of $f$ with prime-to-$p$ branching index), then the stable reduction of $f$ looks very different than what we have in §4.3. In particular, Lemma 4.23 does not hold. In fact, if $\overline{X}_b$ is an étale tail with ramification invariant $\sigma_b = \frac{3}{2}$, then it *must* intersect a $p^2$-component. This is because Lemma 2.14 shows that if $\overline{X}_b$ intersects a $p$-component, we would have to have $\sigma_b = h_b/2$, with $h_b$ prime to 3! With this possibility, many arguments in §4.3 fall apart. In order to repair them, we would need to have good, explicit conditions characterizing the following extensions, in analogy with Corollary 2.34: Say $R$ is a complete discrete valuation ring with fraction field $K$ of characteristic 0, residue field $k$ algebraically closed of characteristic $p$, and uniformizer $\pi$. Suppose further that $R$ contains the $p^2$th roots of unity. Write $A = R\{t\}$ and $L = \text{Frac}(A)$. We wish to characterize $\mathbb{Z}/p^2$-extensions $M$ of $L$ such that the normalization $B$ of $A$ in $M$ satisfies the following conditions:

- Spec $B/\pi \to$ Spec $A/\pi$ is an étale extension with conductor 3.
- Spec $B/\pi$ is integral.

If we allow $p = 5$ the difficulties are less severe. The only places where the assumption $p \neq 5$ matters are for subtle inequalities like $\frac{i-3}{3}(1 + \frac{1}{p-1}) > v(i)$ (see, for instance, the proof of Corollary 4.29), which does not hold for $p = i = 5$, but does hold for any $i > 3$ and $p > 5$. I have worked out computations that fix this problem in most cases, but they are messy and time does not permit their inclusion here.

**Question 5.2.** What can be said if the condition $m = 2$ is relaxed in Theorem 1.4 (ii)?

Relaxing the condition $m = 2$ is more difficult, and I do not necessarily expect to get the same result on the ramification of $p$ in the field of moduli. Allowing arbitrary $m$ will require some new techniques, as the explicit calculations I have done in the $m = 2$ case will no longer work, because the strong auxiliary cover will no longer necessarily be a $p^\nu$-cyclic extension of $\mathbb{P}^1$. I am hopeful that, at least in the case where the deformation data above the original component of $\overline{X}$ are all *multiplicative* (which is true in all of the cases covered in Theorem 1.4), one can obtain results using variations on the deformation theory of torsors under multiplicative group schemes that Wewers develops in [Wew05].

The case where there is an additive deformation datum above the original component of $\overline{X}$ seems more difficult for several reasons. First, the numerical invariants obtained from the deformation data are more flexible in this case, leading to a possibly more complicated-looking stable reduction. In particular, we cannot use Proposition 2.41. Second, Wewers's deformation theory, which depends in an essential way on having a torsor under a multiplicative group scheme, does not apply.

In any case, I would like to have some criteria for determining exactly when the stable reduction of a three-point $G$-Galois cover, where $G$ has a cyclic $p$-Sylow subgroup, will have additive deformation data over the original component. Ideally, these criteria would be based only on the group $G$ and the branching behavior of the cover. Perhaps the criteria will be restrictive enough that the cases where additive deformation data arise can be dealt with on a case-by-case basis.

**Question 5.3.** What happens if we look at three-point $G$-Galois covers where $G$ has a non-cyclic $p$-Sylow subgroup?

The most tractable case here for understanding the ramification of $p$ in the field of moduli is when the $p$-Sylow subgroup is elementary abelian. The most serious difficulty here is that taking the auxiliary cover does not guarantee a simplification of the group-theoretical structure. An advantage, however, is that one can still talk about deformation data, and there will be examples where the deformation data over the original component are all multiplicative. For these examples, it may be possible for the deformation theory of [Wew05] to apply, and we may be able to make some progress.

**Question 5.4.** What happens if we consider covers with four (or more) branch points?

New issues arise here. This is partly because the branch points might collide modulo $p$ and partly because these covers now exist in continuous families, unlike in the case of three-point covers. The case of a four-point cover where the Galois group $G$ has $p$-Sylow subgroup $P$ of order $p$ and $m = |N_G(P)/Z_G(P)| = 2$ has been analyzed to some extent in [BW04]. There the goal is to determine when covers of this form have good reduction. The authors give a lower bound on the ramification of $p$ in the field of moduli in the case where such a cover has bad reduction. This paper does not use deformation data, and I would like to see if combining its techniques with deformation data can yield results giving an upper bound on the ramification of $p$ in the field of moduli. Perhaps we can even replace the assumption $m = 2$ by some weaker assumption involving multiplicative reduction.

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