Liberian Mathematics Teacher Training Program 2023–2024

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1. Suppose a regular hexagon has side length s. What is its area?
2. Compare your answer above to the area of an equilateral triangle with side length s. How many times bigger is the hexagon's area? Can you give a geometric reason why this should be true?

Area: $\dfrac{ns^2}{4\tan(180°/n)}$.

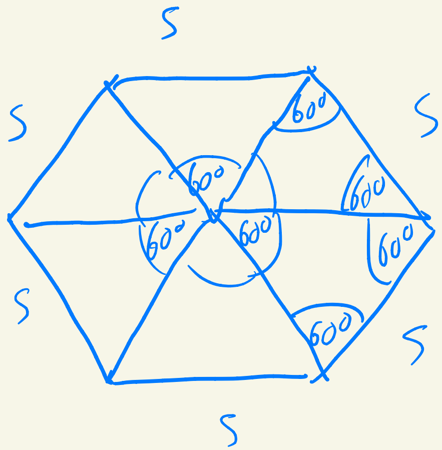
1) $n=6$: $\dfrac{6\cdot s^2}{4\tan(\frac{180°}{6})} = \dfrac{6\cdot s^2}{4\tan(30°)} = \dfrac{6s^2}{4\cdot\frac{\sqrt{3}}{3}} = \dfrac{6s^2}{4}\cdot\dfrac{3}{\sqrt{3}} = \dfrac{18s^2}{4\sqrt{3}} = \boxed{\dfrac{9s^2}{2\sqrt{3}}}$

$= \dfrac{3\cdot 3s^2}{2\sqrt{3}} = \boxed{\dfrac{3\sqrt{3}s^2}{2}}$

2) $n=3$: Last time: $\dfrac{s^2\sqrt{3}}{4}$

Ratio: $\dfrac{\text{Hexagon area}}{\text{Triangle area}} = \dfrac{3\cancel{\sqrt{3}}\cancel{s^2}/2}{\cancel{s^2}\cancel{\sqrt{3}}/4} = \dfrac{3/2}{1/4} = \dfrac{3}{2}\cdot 4 = \dfrac{12}{2} = \boxed{6}$

Hexagon area is 6 times triangle area.



Interior
Angles in hexagon : 120°

Hexagon is made out of 6 copies of the equilateral triangle!

Recall: area of a regular polygon

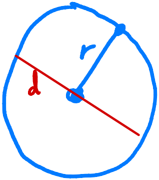
Recall that the area of a regular polygon with n sides of length s is

$$\frac{ns^2}{4\tan(180^\circ/n)}.$$

Circle basics

- A *circle* is the set of points in the plane at a given distance from a fixed point, called the *center*.
- The distance from the center to any point on the circle is called the *radius*.
- The length of a line segment from one point on the circle to another passing through the center is called the *diameter*.
- The "perimeter" of the circle is also known as the *circumference*.

$d = 2r$



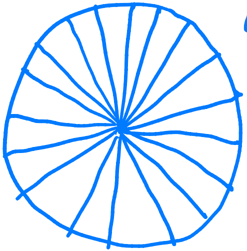
Basic formulas for circles

- If a circle has radius r, its circumference c is given by $2\pi r$.
- This is not so much a formula to be proved as it is a *definition* of π.
- It says that, not matter what circle you draw, the ratio $c/2r$ will be the same. This number is what we call π.
- The *area* of a circle with radius r is given by πr^2.

$$c = 2\pi r \iff \pi = \frac{c}{2r}$$

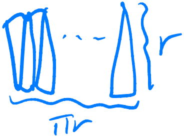
Justification for area formula

The following picture is a “proof” of the area formula for a circle. A rigorous proof requires calculus. In fact, you need calculus even to give an accurate *definition* of what the area of a circle means.



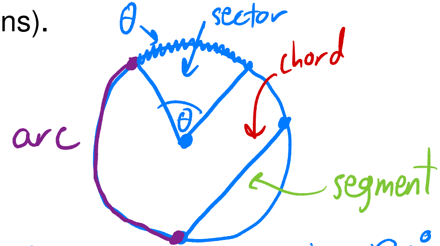
Each piece is close to a triangle.

→ rearrange:



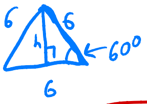
Other parts of circles

- A *sector* is the part of a circle between two line segments from the center to the circle.
- A *chord* is a line segment drawn from one point on a circle to another.
- A *segment* is the part of a circle lying on one side of a chord.
- An *arc* is a connected section of a circle. It can be measured in degrees (or radians).



Full circle $= 360^\circ$ of arc. Semicircle $= 180^\circ$, etc.

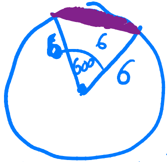
What is the area of the sector of a circle of radius 6 cut out by two radii at a 60° angle? What about the corresponding segment shown in the diagram below?



$\frac{h}{6} = \sin 60° = \frac{\sqrt{3}}{2}$

$\rightsquigarrow 2h = 6\sqrt{3}$

$h = 3\sqrt{3}.$



Area: $\frac{1}{2} \cdot 6 \cdot 3\sqrt{3}$
$= 9\sqrt{3}.$

$r = \frac{1}{2} d$

$r = \frac{C}{2\pi}$

Sector area: Whole circle has area $\pi r^2 = \pi (6)^2 = 36\pi$

Sector area $=$ circle area $\times \underbrace{\frac{\cancel{60}\,1}{\cancel{360}\,6}}_{\text{sector proportion}} = 36\pi \cdot \frac{1}{6} = \boxed{6\pi}$

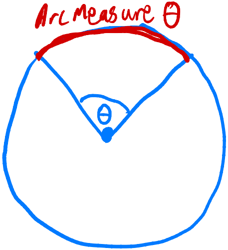
Segment area: Sector area − triangle area.

$= 6\pi - \frac{6^2\sqrt{3}}{4} = 6\pi - \frac{36\sqrt{3}}{4} = \boxed{6\pi - 9\sqrt{3}} \approx 3.26$

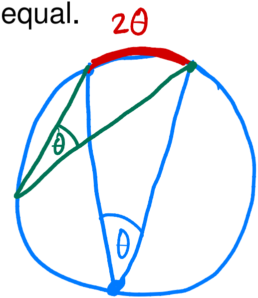
Angles inside circles: Central angles

A *central* angle is an angle formed by two radii of a circle. Its measure is the same as the measure of the arc it cuts out.

"subtends"

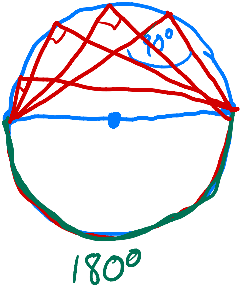


- An inscribed angle is an angle formed by two chords of a circle meeting at a point on the circle. Its measure is *half* the measure of the arc it cuts out.
- A key consequence of this is that any two incribed angles cutting out the same arc are equal.



Angle example

Let's consider an angle inscribed in a semicircle:

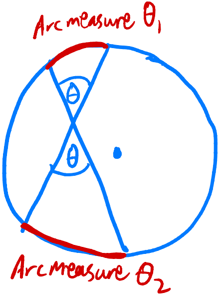


upshot: Every angle inscribed in a semicircle is a right angle!

Exactly 90°.

Angles inside circles: Angles formed by crossing lines

If two chords meet in a circle, the angle that they form is the average of the arc measure cut out by the angle and the arc measure cut out by the vertical angle.

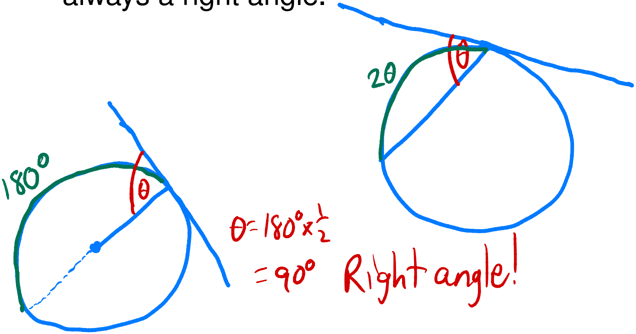


Formula: $\theta = \dfrac{\theta_1 + \theta_2}{2}$.

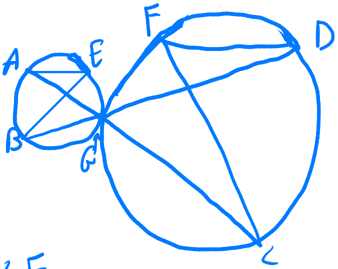
→ Example: If $\theta_1 = 30°$, $\theta_2 = 50°$, then $\theta = \dfrac{30° + 50°}{2} = 40°$.

Angles inside circles: Angles formed using a tangent line

- The angle formed by a *tangent* line to a circle and a chord of that circle is half the arc measure cut out by the chord (on the relevant side of the tangent line).
- In particular, the angle between a radius and a tangent line is always a right angle.



An angle proof



inscribed ∠'s w/same arc

Q: Show $\angle E = \angle F$

A: $\angle E = \angle AGB = \angle DGC = \angle F$

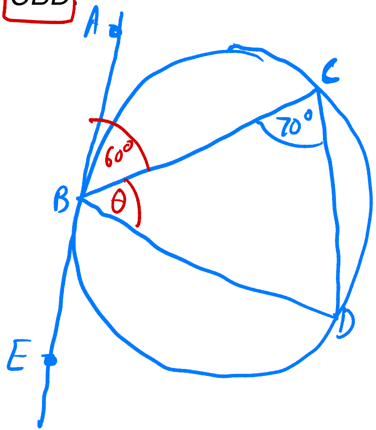
vertical angles

inscribed ∠'s w/same arc

1. What is the circumference of a circle whose area is 8π?
2. In the figure below, given that angle ABC is 60° and angle BCD is 70°, find angle $\boxed{CBD}$.

$\angle ABC = 60^\circ$

$\angle CBD = \theta$



Thank you for your attention! Next week, on January 26, we will discuss circles in more depth.

Preview: Class Jan. 26 } Finish
Feb. 2 } Geometry unit

No class Feb. 9

Class Feb. 16 ----} New unit (input from participants)

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until May.