Liberian Mathematics Teacher Training Program 2023–2024

Andrew Obus[1]

CUNY Baruch College

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HW Exercise 1

What is the circumference of a circle whose area is 8π?

$8\pi = \pi r^2$ (Area)

$\div \pi \quad \pi$

$8 = r^2$

$r = \sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$

Circumference: $C = 2r\pi$

$= 2 \cdot 2\sqrt{2} \cdot \pi$

$= \boxed{4\sqrt{2} \cdot \pi}$

≈ 17.77

$\pi \approx 22/7$

$\pi = 3.14159\ldots$

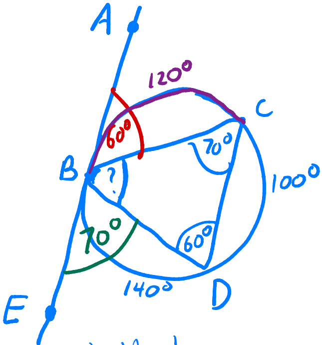
$22/7 = 3.142857\ldots$

3.14 is a better approxi to π than $22/7$.

HW Exercise 2

In the figure below, given that angle ABC is 60° and angle BCD is 70°, find angle CBD.

$\angle DBE = \angle DCB$
(cut out same arc!)
Since $\angle DCB = 70^\circ$,
we have
$\angle DBE = 70^\circ$



So $\angle CBD = \dfrac{100^\circ}{2}$
$= 50^\circ$

Note: $\overline{ABE}$ is a straight line!
So $60^\circ + \angle CBD + 70^\circ = 180^\circ \rightsquigarrow \boxed{\angle CBD = 50^\circ}$

Recall: Inscribed angles in circles

- An inscribed angle is an angle formed by two chords of a circle meeting at a point on the circle. Its measure is *half* the measure of the arc it cuts out.
- A key consequence of this is that any two inscribed angles cutting out the same arc are equal.

Recall: Chord-tangent angles

- The angle formed by a *tangent* line to a circle and a chord of that circle is half the arc measure cut out by the chord (on the relevant side of the tangent line).
- In particular, the angle between a radius and a tangent line is always a right angle.

Proof of the inscribed angle formula

Want to show:

$\theta = \frac{1}{2}\phi$, or

$2\theta = \phi$.

$\triangle ABC$

Equations:

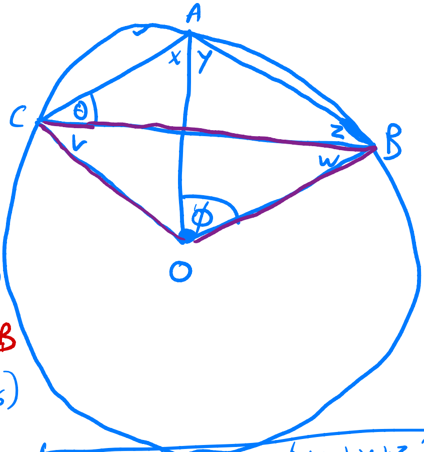
$\theta + x + y + z = 180^\circ$ (1)

$\phi + w + y + z = 180^\circ$ (2)

$\triangle AOB$

$v = w$ (3) ($\triangle BOC$ is isosceles)

$x = \theta + v$ (4) ($\triangle AOC$ is isosceles).



observe:

$\overline{OA} = \overline{OB} = \overline{OC}$

Subbing (3) into (2): $\phi + v + y + z = 180^\circ$

" (4) into (1): $\theta + (\theta + v) + y + z = 180^\circ$

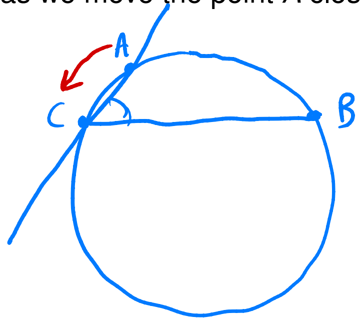
$2\theta + v + y + z = 180^\circ$.

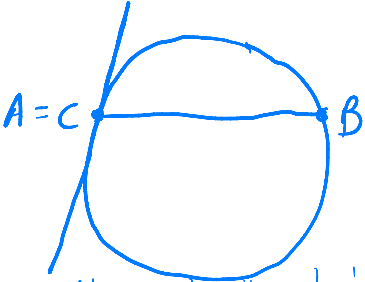
Subtract: $\phi - 2\theta = 0^\circ$, or $\boxed{\phi = 2\theta}$

Chord-tangent angles as limits of inscribed angles

Consider the inscribed angle *ACB* in the circle below. What happens as we move the point *A* closer to the point *C*?

Know: $\angle ACB = \frac{1}{2}\,\text{measure}(\widehat{AB})$





As A moves closer to C, the line through A and C becomes the tangent line through C.

At every step, the angle is half the arc. This remains true when A gets all the way to C.

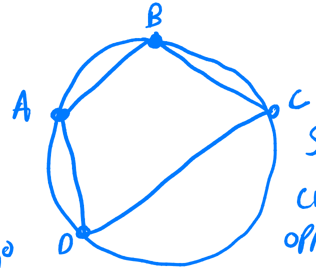
A consequence of the inscribed angle formula: Cyclic quadrilaterals

- We have seen that every regular polygon has a circumscribed circle.
- In fact, *every* triangle has a circumscribed circle (this requires proof!).
- However, it is *not* true that every quadrilateral has a circumscribed circle. Why not?

$\angle D = \frac{1}{2}\,\overset{\frown}{ABC}$.

$\angle B = \frac{1}{2}\,\overset{\frown}{ADC}$.

So $\angle D + \angle B = \frac{1}{2}(\overset{\frown}{ABC} + \overset{\frown}{ADC}) = \frac{1}{2}(360^\circ) = 180^\circ$

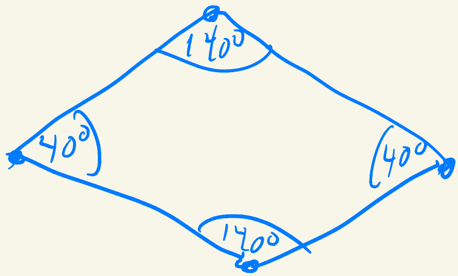


So $\angle B + \angle D = 180^\circ$.

Similarly: $\angle A + \angle C = 180^\circ$.

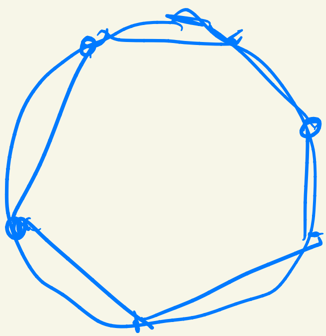
So a quadrilateral w/ a circumscribed circle has opposite angles adding to 180°.

Example:

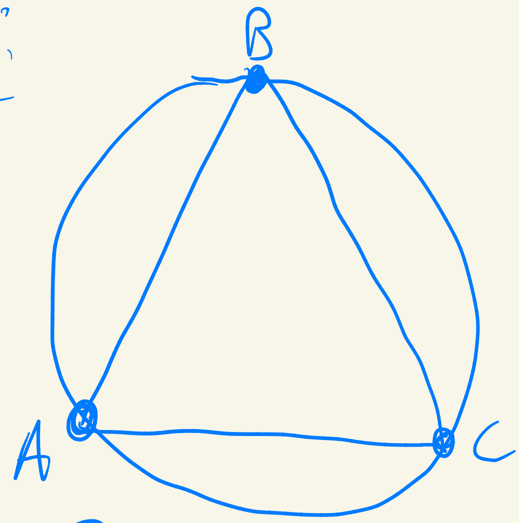


This quadrilateral is not cyclic.

$$\left(\begin{array}{l} 140^\circ + 140^\circ \neq 180^\circ \\ 40^\circ + 40^\circ \neq 180^\circ \end{array}\right)$$



Triangles:

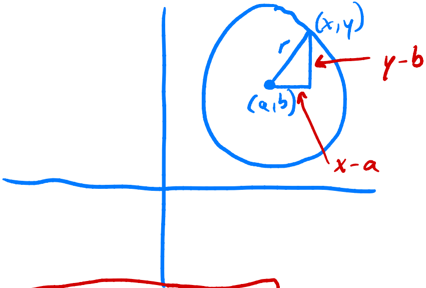


$$\angle A = \tfrac{1}{2}\widehat{BC}$$
$$\angle B = \tfrac{1}{2}\widehat{AC}$$
$$+\quad \angle C = \tfrac{1}{2}\widehat{AB}$$

$$\angle A + \angle B + \angle C = \tfrac{1}{2}\left(\widehat{BC} + \widehat{AC} + \widehat{AB}\right)$$
$$= \tfrac{1}{2}(360^\circ)$$
$$= 180^\circ$$

Equations for circles

Let's consider a circle in the plane with center at the point (a, b) and radius r. What can we say about a point (x, y) that lies on the circle?



Pythagoras: $\boxed{(x-a)^2+(y-b)^2=r^2}$

Ex: Circle: Radius 3, center $(1,2)$: $(x-1)^2+(y-2)^2=9$

Recall: Completing the square

- Suppose you want to solve the quadratic equation $x^2 + 10x + 21 = 0$.
- One way to do this is to write $x^2 + 10x + 21$ as $x^2 + 10x + 25 - 4$, or $(x + 5)^2 - 4$.
- Then, solving the equation $(x + 5)^2 - 4 = 0$ is easy — you just write $(x + 5)^2 = 4$, so $x + 5 = \pm 2$.
- This yields $x = -7$ or $x = -3$.

(Exc. Same answer as factoring gives).

Example

- The technique of completing the square can help us extract information from the equation of a circle.
- The equation $x^2 + y^2 + 4x + 6y = 29$ is the equation of a circle. What are its center and radius?

Complete the square:

$$x^2+4x+4+y^2+6y+9=29+4+9$$

$$\to (x+2)^2+(y+3)^2=42$$

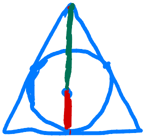
$$\to (x-(-2))^2+(y-(-3))^2=42$$

Center: $\boxed{(-2,-3)}$ Radius: $\boxed{\sqrt{42}.}$

Homework Exercises

1. Find the center and radius of the circle given by the equation $x^2 + y^2 + 8x - y = 20$.
2. Suppose a circle is inscribed in an equilateral triangle. What is the ratio of the circle's area to that of the triangle. **Hint**: You may use the fact that the radius of the circle is one-third the altitude of the triangle, as shown below.
3. Now suppose a circle is inscribed in a square. What is the ratio of the circle's area to that of the square? (This is a bit easier than the previous problem). Is this ratio larger or smaller than in the case of a triangle? Does this suggest a pattern to you?

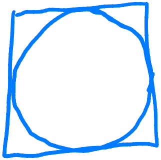
2)



Can use!

$\text{red} = \frac{1}{3}(\text{red} + \text{green})$

3)



Thank you for your attention! Next week, on February 2, we will finish up our geometry unit.

No Class Feb. 9.

Resume Feb. 16.