Liberian Mathematics Teacher Training Program 2023–2024

Andrew Obus[1]

CUNY Baruch College

February 2, 2024

[1]This program is partially supported by NSF CAREER Grant DMS-2047638

HW Exercise 1

Find the center and radius of the circle given by the equation $x^2 + y^2 + 8x - y = 20$.

Recall:
$(x-a)^2+(y-b)^2=r^2$
Center: (a,b)
Radius: r

$x^2+8x \quad + y^2 - y \quad = 20$

$\div 2 \downarrow \quad 4 \qquad \div 2 \downarrow \quad \frac{1}{2}$

$(x+4)^2 - 16 + (y-\frac{1}{2})^2 - \frac{1}{4} = 20$

$\Rightarrow (x+4)^2 + (y-\frac{1}{2})^2 = 20+16+1/4 = 36\,1/4 = \frac{145}{4}$

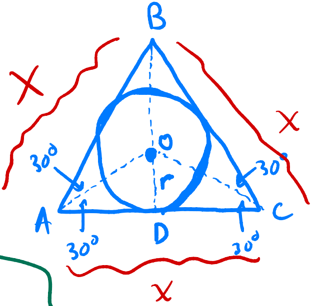
$(x-(-4))$

Center: $\boxed{(-4, \frac{1}{2})}$

Radius: $\sqrt{\frac{145}{4}} = \boxed{\frac{\sqrt{145}}{2}} \approx 6.02$

HW Exercise 2

Suppose a circle is inscribed in an equilateral triangle. What is the ratio of the circle's area to that of the triangle? **Hint**: You may use the fact that the radius of the circle is one-third the altitude of the triangle, as shown below.



Ratio:

$$\frac{\text{Area of Circle}}{\text{Area of triangle}} = \frac{\frac{\pi}{12}\cancel{x^2}}{\frac{\sqrt{3}}{4}\cancel{x^2}} = \frac{\pi}{\cancel{12}_3}\cdot\frac{\cancel{4}^1}{\sqrt{3}} = \boxed{\frac{\pi}{3\sqrt{3}}} \approx .6046$$

Area of triangle:

$$h = \frac{\text{Altitude}}{x} = \sin 60^\circ = \frac{\sqrt{3}}{2} \Rightarrow h = \frac{\sqrt{3}}{2}x$$

$$\text{So area} = \frac{1}{2}xh = \boxed{\frac{\sqrt{3}}{4}\cdot x^2}$$

Area of Circle:

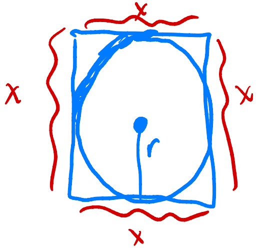
$$\text{Radius: } \frac{r}{AD} = \tan 30^\circ$$

$$\Rightarrow \frac{r}{x/2} = \frac{\sqrt{3}}{3} \Rightarrow r = \frac{\sqrt{3}}{3}\cdot\frac{x}{2} = \frac{\sqrt{3}}{6}x$$

$$\text{Area: } \pi r^2 = \pi\left(\frac{\sqrt{3}}{6}x\right)^2 = \pi\frac{3}{36}x^2 = \boxed{\frac{\pi}{12}\cdot x^2}$$

HW Exercise 3

Now suppose a circle is inscribed in a square. What is the ratio of the circle's area to that of the square? (This is a bit easier than the previous problem). Is this ratio larger or smaller than in the case of a triangle? Does this suggest a pattern to you?



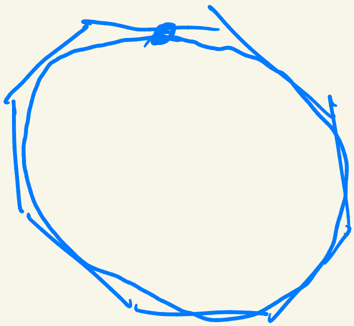
Area of square: x^2

Area of circle

$r = x/2$

Area: $\pi r^2 = \pi (x/2)^2 = \frac{\pi x^2}{4}$.

Ratio: $\frac{\pi \cancel{x^2}/4}{\cancel{x^2}} = \boxed{\pi/4} \approx .785 \rightarrow$ Bigger!

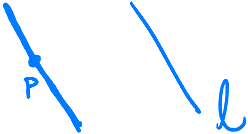


← Circle is a large part of the area!

The more sides in a regular polygon, the larger proportion of the area the circle takes up.

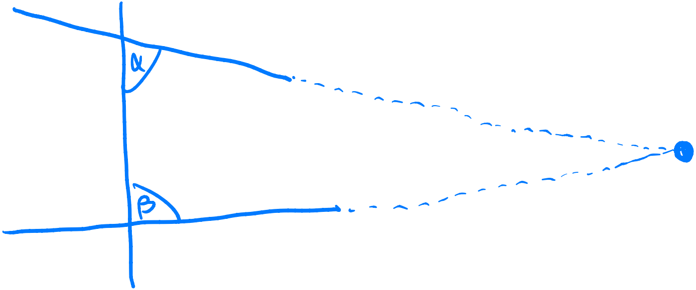
Euclidean geometry: the parallel postulate

- At the beginning of our geometry unit, I mentioned that Euclid's approach to geometry was to begin with *axioms* (or *postulates*), and to derive all results from the axioms using proofs.
- One of the most important of Euclid's postulates is the so-called "parallel postulate" or "fifth postulate". It states: given a line ℓ in the plane, and a point P not on the line, there exists exactly one line passing through P that is parallel to ℓ.
- Technically, this is called "Playfair's postulate". In fact, Euclid stated this postulate in a different, but equivalent way (but it is not obvious why they are equivalent)!
- For centuries, mathematicians tried to prove this postulate from the other ones, but none succeeded.



Euclid’s fifth postulate vs. Playfair’s postulate

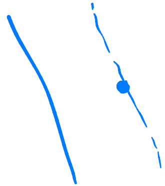
Euclid’s fifth postulate states that if two lines cross a given line at distinct points, and if they make angles on the same side of the line adding up to less than 180°, then the two lines meet.



$\alpha + \beta < 180^\circ$

Non-euclidean geometry

- Playfair's postulate seems "obvious". But what if we discard it?
- Around 1830, Bolyai and Lobachevsky discovered what happens if you replace Playfair's postulate with one stating that there is *more than one* line passing through P that is parallel to ℓ. This is called "hyperbolic geometry".
- Around the same time, it was also discovered what happens if you replace Playfair's postulate with one stating that there is *no* line passing through P that is parallel to ℓ. This is called "elliptic geometry" or "spherical geometry".

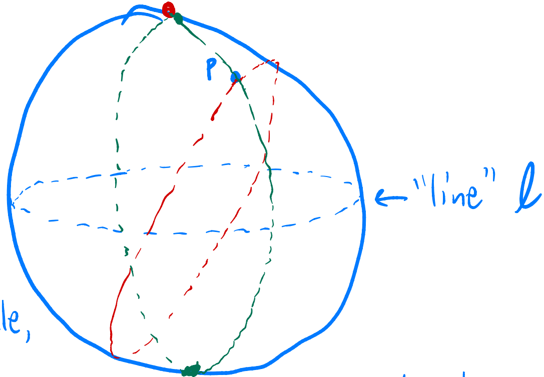


Elliptic geometry

- Elliptic geometry is maybe easier to understand than hyperbolic geometry.
- For instance, we can think of it as geometry on the surface of the earth.
- “Lines” in this geometry are great circles (i.e., circles on ~~the~~ the face of the earth that break it up into two equal halves). e.g, equator
- Any two such circles meet! For instance, the equator intersects all meridians.
- Furthermore, triangles in elliptic geometry have angles that sum to *more* than 180°!

Failure of Playfair's postulate in elliptic geometry

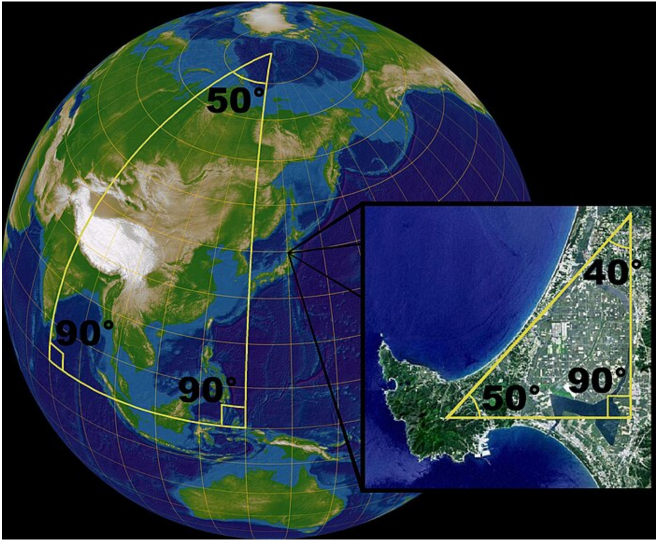
Any two great circles meet.



So given a great circle and a point not on the circle, P

then all great circles through P meet the original great circle!

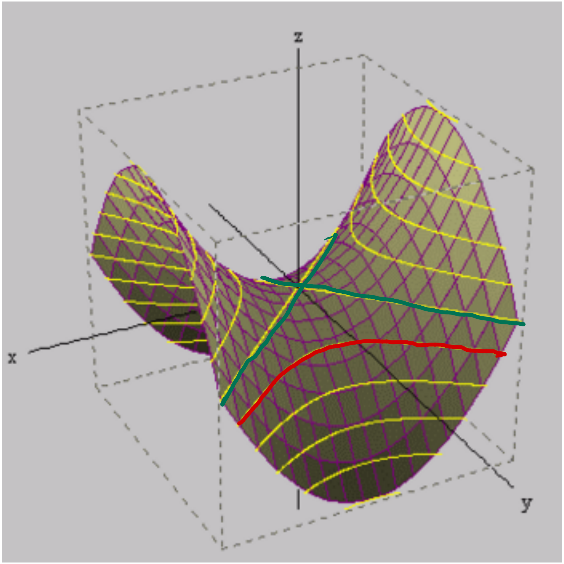
Picture of a triangle in elliptic geometry



Hyperbolic geometry

- We can think of hyperbolic geometry as geometry on a "saddle-like" surface.
- "Lines" in this geometry are a bit harder to describe — but we can think of the line between two points as the shortest path from one point to the other point.
- As we will see in the following pictures, you can have many lines through one point that are parallel to a given line.
- Also, triangles in hyperbolic geometry have angles that sum to *less* than 180°!

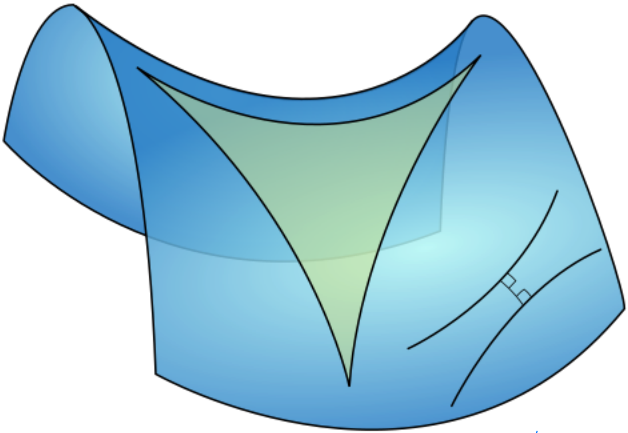
Failure of Playfair's postulate in hyperbolic geometry



$x^2 - y^2 = z$

or $z = xy$

Picture of a triangle in hyperbolic geometry



Angles add up to less than 180°.

Thank you for your attention! There is no class next week, February 9. We will resume on February 16.

There is no standard homework this week, but I will send out a survey polling you about what topics you are interested in covering in the spring.

→ Proof-based euclidean geometry

- Trigonometry
- Conic Sections
- Logic
- ....