Liberian Mathematics Teacher Training Program 2023–2024

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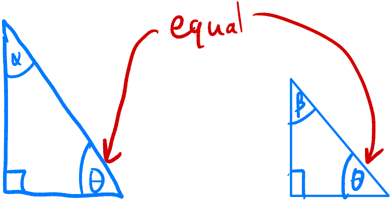
Introduction to trigonometry: Right Triangles

- All right triangles share one common angle measure. So to show that two right triangles are similar, it is enough to show that the measure of *one* non-right angle in one triangle equals that of a non-right angle in the other triangle.
- This means that if we have two right triangles with a common non-right angle measure, the ratios of their corresponding sides are the same.
- This is why it makes sense to define the *trigonometric functions*.

$\alpha + 90^\circ + \theta = 180^\circ$

$\rightsquigarrow \alpha = 180^\circ - 90^\circ - \theta$



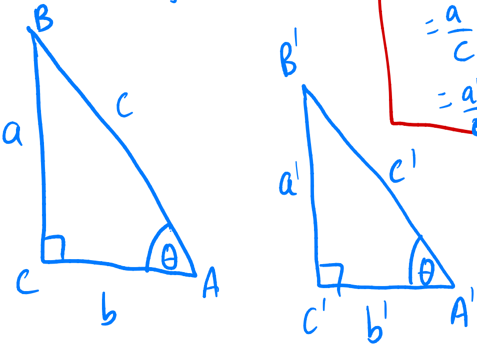
This implies that the triangles are similar.

$\beta = 180^\circ - 90^\circ - \theta$

ONLY POSSIBILITY: $\alpha = \beta$.

Recall: Ratios of sides in similar triangles

2 similar triangles:



Want: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$= \dfrac{a}{c}$ (1^{st} Δ)

$= \dfrac{a'}{c'}$ (2^{nd} Δ)

Common Ratios: $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

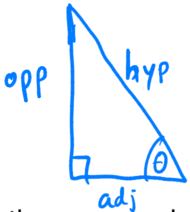
Consequences: $\dfrac{a}{a'} = \dfrac{c}{c'} \Rightarrow ac' = ca'$

$\Rightarrow \boxed{\dfrac{c'}{a'} = \dfrac{c}{a}}$

Similarly: $\dfrac{b'}{a'} = \dfrac{b}{a}$, $\dfrac{c'}{b'} = \dfrac{c}{b}$ (Exc.)

Trigonometric functions

- From now on, we will not distinguish between angles and their measures. Let θ be an angle such that $0^\circ < \theta < 90^\circ$.
- Draw a right triangle with one angle θ. Label the sides of the triangle “opp”, “adj”, and “hyp”.
- We define the following trigonometric functions of θ:
  - $\sin\theta = \text{opp}/\text{hyp}$
  - $\cos\theta = \text{adj}/\text{hyp}$
  - $\tan\theta = \text{opp}/\text{adj}$
  - $\cot\theta = \text{adj}/\text{opp}$
  - $\sec\theta = \text{hyp}/\text{adj}$
  - $\csc\theta = \text{hyp}/\text{opp}$



- The first three can be remembered with the mnemonic “SOHCAHTOA”.
- Because of similar triangles, these are well-defined functions!

Basic relationships between trigonometric functions

- $\overset{\text{cot}}{\cancel{\tan}}\,\theta = 1/\overset{\text{tan}}{\cancel{\cot}}\,\theta$
- $\csc\theta = 1/\sin\theta$
- $\sec\theta = 1/\cos\theta$

sine, cosine, tangent = "basic"

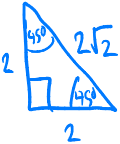
secant, cosecant, cotangent = "reciprocal"

WORTH MEMORIZING!!!

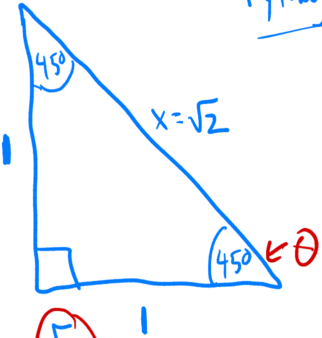
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	1	0	undefined

Pattern for sine column: Entries: $\underset{\underset{0}{\shortparallel}}{\frac{\sqrt{0}}{2}}, \underset{\underset{\frac{1}{2}}{\shortparallel}}{\frac{\sqrt{1}}{2}}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \underset{\underset{1}{\shortparallel}}{\frac{\sqrt{4}}{2}}.$

The 45-45-90 triangle



$$\sin 45^\circ = \frac{\cancel{2}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$



Pythagoras: $1^2 + 1^2 = x^2$

OR $2 = x^2$

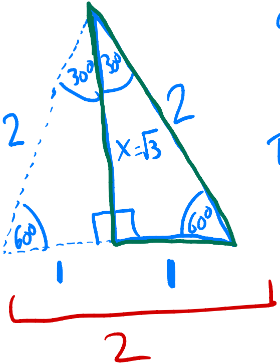
OR $x = \sqrt{2}$.

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1.$$

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$



Observe: All angles of big triangle are $60°$!

That means the triangle is equilateral.

To find height x: Pythagoras: $1^2 + x^2 = 2^2 \rightsquigarrow x^2 = 2^2 - 1^2$

$x^2 = 3$

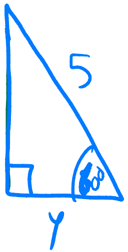
$x = \sqrt{3}$.

$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$ $\quad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$

$\tan 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{1} = \sqrt{3}$.

Fill in the remaining sides of the following right triangle:

$\frac{5\sqrt{3}}{2} = x$



Use trigonometry

$\sin 60^\circ = \frac{x}{5}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{x}{5}$

cross multiply

$\Rightarrow 5\sqrt{3} = 2x$

$\Rightarrow \boxed{\frac{5\sqrt{3}}{2} = x}$

$\cos 60^\circ = \frac{y}{5}$

$\Rightarrow \frac{1}{2} = \frac{y}{5}$

cross-multiply

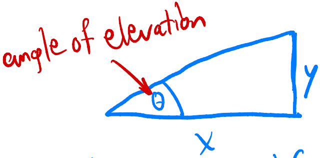
$\Rightarrow 5 = 2y$

$\Rightarrow \boxed{y = \frac{5}{2}}$

Percentage grade

- In road construction, an important quantity is the *percentage grade*, which represents the steepness of the road.
- By definition, the percentage grade is the ratio of vertical rise or fall to horizontal distance covered, expressed as a percent.
- On the other hand, the *angle of elevation* of a road is the angle that road makes with respect to an imaginary horizontal line on the earth's surface.

$\%$ Grade $\stackrel{\text{def}}{=} \frac{y}{x}$ (as a percent).



Formula: $\%$ grade = tangent(angle of elevation).

An angle of elevation example

Suppose a road has a 5% grade. What is its angle of elevation?

Let θ = angle of elevation.

Solve $\tan(\theta) = .05$

$$\theta = \tan^{-1} .05 \approx \boxed{2.86^\circ}$$

A more complicated angle of elevation example

Two observers are looking at a tree in the distance, one standing behind the other one as they look at the tree. One measures the angle of elevation of the tree to be 30°. The other measures the angle of elevation to be 20°. If the distance between the two observers is 50 feet, then how tall is the tree?

Step 1: Draw diagram.

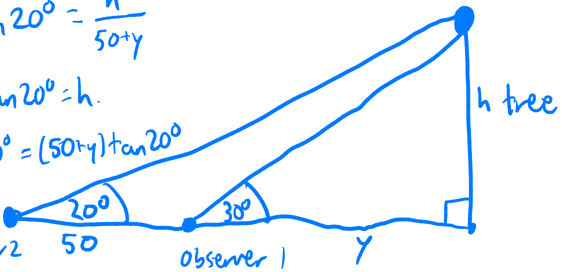
Step 2: Write down trig equation for each right triangle.

(1) $\tan 30^\circ = \frac{h}{y}$ (2) $\tan 20^\circ = \frac{h}{50+y}$

OR $y \tan 30^\circ = h$ $(50+y)\tan 20^\circ = h$.

Step 3: Eliminate h: $y\tan 30^\circ = (50+y)\tan 20^\circ$

solve for y



$$y\tan 30^\circ = (50+y)\tan 20^\circ$$
$$= 50\tan 20^\circ + y\tan 20^\circ$$

$$\Rightarrow y\tan 30^\circ - y\tan 20^\circ = 50\tan 20^\circ$$

$$\Rightarrow y(\tan 30^\circ - \tan 20^\circ) = 50\tan 20^\circ$$

$$\Rightarrow y = \frac{50\tan 20^\circ}{\tan 30^\circ - \tan 20^\circ} \approx 85.29$$

4 Solve for h.

$y\tan 30^\circ = h$. $\quad y = 85.29$

So $h = (85.29)\cdot\tan 30^\circ$

$$= 85.29 \cdot \frac{\sqrt{3}}{3} \approx \boxed{49.24 \text{ feet}}$$

Homework Exercises

1. Prove that if $0^\circ < \theta < 90^\circ$, then $\sin\theta = \cos(90^\circ - \theta)$.
2. Suppose a right triangle has hypotenuse of length 5 and one angle measuring 37°. What are the lengths of the other sides of the triangle, to the nearest tenth? (this requires a calculator)
3. Two observers spaced 400 feet apart look up at a hot air balloon flying *between* them. One measures the angle of elevation at 40°, and the other measures it at 50°. How high is the balloon flying? *Hint:* Draw the diagram carefully! You may want to label the distance on the ground from one observer to the point directly below the balloon.

Thank you for your attention! Next week, on February 23, we will discuss the unit circle. → Also more angle of elevation and angle of depression.